Solow Growth Model

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Basic Solow Model

Solow (1956, 1970) introduced a model of economic growth

- Basis for most growth theory, for Real Business Cycle models and New Keynesian modeling
- several very specific and testable results about growth

Kydland and Prescott (1982)

- Stochastic version to study business cycles
    - Real Business Cycle theory
    - New Keynesian models

Basic Solow Model

Solow's model:

- CRS production function, law of motion for capital and savings rate
- Equilibrium conditions: Investment = Savings

From this simple model

- First-order difference equation
    - evolution of capital per worker
- Time path of the economy

The Basic Model

The production function

$$Y_t = A_t F(K_t, H_t)$$

Y_t: output of the single good in the economy at date t

A_t: level of technology

- $A_t = (1+\alpha)^t A_0$, where A_0 is the time 0 level of technology and α is the net rate of growth of technology

K_t: capital stock

H_t: quantity of labor used in production

$F(K_t, H_t)$: homogeneous of degree one

The Basic Model

Output per worker: $y_t = Y_t / H_t$.

$$y_t = \frac{Y_t}{H_t} = A_t F\left(\frac{K_t}{H_t}, \frac{H_t}{H_t}\right) = A_t F(k_t, 1) = A_t f(k_t)$$

k_t: capital per worker

The Basic Model

Labor force grows at a constant net rate n

$$H_{t+1} = (1+n)H_t$$

Capital law of motion

$$K_{t+1} = (1-\delta)K_t + I_t$$

δ: the depreciation rate
I_t: investment at time t

The Basic Model

Capital per worker grow according to the rule

$$k_{t+1} = \frac{(1-\delta)k_t + i_t}{1+n}$$

Savings: fixed fraction σ of output

$$s_t = \sigma y_t$$

Equilibrium in a closed economy:: $i_t = s_t$

The Basic Model

Capital difference equation:

$$(1+n)k_{t+1} = (1-\delta)k_t + \sigma(1+\alpha)^t A_0 f(k_t)$$

Simplest case: technology growth is zero ($\alpha = 0$)

$$(1+n)k_{t+1} = (1-\delta)k_t + \sigma A_0 f(k_t)$$

Stationary state: $k_{t+1} = k_t = \bar{k}$

$$(1+n)\bar{k} = (1-\delta)\bar{k} + \sigma A_0 f(\bar{k})$$

The Basic Model

Stationary state: $k_{t+1} = k_t = \bar{k}$

$$(1+n)\bar{k} = (1-\delta)\bar{k} + \sigma A_0 f(\bar{k})$$

or when

$$(\delta+n)\bar{k} = \sigma A_0 f(\bar{k})$$

- There is a stationary state at $\bar{k} = 0$ and one for a positive $\bar{k}$.
- All economies with $k_0 \neq 0$ converge to the $+$ stationary state.

The Basic Model

Stationary state: $k_{t+1} = k_t = \bar{k}$

$$(1+n)\bar{k} \quad = \quad (1-\delta)\bar{k} + \sigma A_0 f(\bar{k})$$

or when

$$(\delta+n)\bar{k} \quad = \quad \sigma A_0 f(\bar{k})$$

Stability conditions of the positive stationary state

$$k_{t+1} \quad = g(k_t) = \quad \frac{(1-\delta)k_t + \sigma A_0 f(k_t)}{1+n} \tag{1}$$

The Basic Model

Important results of the Solow growth model

- If all economies have access to the same technology, poorer ones (those with less initial capital) will grow faster than richer ones (those with more initial capital).

Let $\gamma_t = k_{t+1}/k_t$

$$\gamma_t = \frac{k_{t+1}}{k_t} \quad = \quad \frac{(1-\delta)k_t + \sigma A_0 f(k_t)}{(1+n)k_t}$$

Take the derivative of γ_t with respect to k_t to see how the growth rate depends on the initial capital stock.

The Basic Model

$$\gamma_t = \frac{k_{t+1}}{k_t} = \frac{(1-\delta)k_t + \sigma A_0 f(k_t)}{(1+n)k_t}$$

Take the derivative of γ_t with respect to k_t to see how the growth rate depends on the initial capital stock

$$\frac{d\gamma_t}{dk_t} = \frac{\sigma A_0}{(1+n)k_t^2}\left[f'(k_t)k_t - f(k_t)\right]$$

which is negative when $k_t > 0$.

- Growth rate of capital declines as the capital stock increases

The Basic Model

Main results of the simplest version of the Solow model

When all countries have access to the same technology and all have the same savings rate

- all countries converge to the same levels of capital and output per worker and
- poorer countries grow faster than richer ones

Technological Growth

With a constant rate of technological growth, all economies converge toward a **Balanced Growth Path (BGP)**

- **BGP**: growth rate of capital and output are constant

Cobb-Douglas production function $f(k_t) = k_t^\theta$

- θ: fraction of the economy's income that goes to capital.

We are looking for an equilibrium where capital grows at some unknown constant rate $\gamma = k_{t+1}/k_t$

Technological Growth

Normalize initial technological level $A_0 = 1$

$$\gamma = \frac{k_{t+1}}{k_t} = \frac{(1-\delta)k_t + \sigma(1+\alpha)^t k_t^{\theta}}{(1+n)k_t} = \frac{(1-\delta)}{(1+n)} + \frac{\sigma(1+\alpha)^t}{(1+n)k_t^{1-\theta}}$$

Rearranging....

$$k_t = \left[\frac{\sigma(1+\alpha)^t}{(1+n)\gamma-(1-\delta)}\right]^{\frac{1}{1-\theta}} = (1+\alpha)^{\frac{t}{1-\theta}}\left[\frac{\sigma}{(1+n)\gamma-(1-\delta)}\right]^{\frac{1}{1-\theta}}$$

Technological Growth

Along a balanced growth path, the constant growth rate of capital per worker γ must be equal to

$$\lambda = \frac{k_{t+1}}{k_t} = \frac{(1+\alpha)^{\frac{t+1}{1-\theta}} \left[\frac{\sigma}{(1+n)\gamma-(1-\delta)}\right]^{\frac{1}{1-\theta}}}{(1+\alpha)^{\frac{t}{1-\theta}} \left[\frac{\sigma}{(1+n)\gamma-(1-\delta)}\right]^{\frac{1}{1-\theta}}} = (1+\alpha)^{\frac{1}{1-\theta}}$$

And along this path, output per worker (also) grows by

$$\frac{y_{t+1}}{y_t} = (1+\alpha)^{\frac{1}{1-\theta}} = \lambda$$

The Golden Rule

Welfare: function of only consumption

- Maximize welfare in a stationary state - maximize the steady state level of consumption

The savings rate that maximizes consumption:

- The **golden rule** saving rate

Since production can be either saved (invested) or consumed, the per worker consumption is

$$c_t \quad = (1-\sigma)y_t \quad = (1-\sigma)A_0 f(\bar{k})$$

The Golden Rule

Per worker consumption

$$c_t = (1-\sigma) y_t = (1-\sigma) A_0 f(\bar{k})$$

For an economy without technological growth, the condition for a stationary state is

$$(\delta + n)\bar{k} = \sigma A_0 f(\bar{k})$$

Substituting the condition for the stationary state into the consumption equation

$$\bar{c} = A_0 f(\bar{k}) - (\delta + n)\bar{k}$$

The Golden Rule

$$\bar{c} = A_0 f(\bar{k}) - (\delta + n)\bar{k}$$

First-order conditions for maximizing consumption give

$$A_0 f'(\bar{k}) = (\delta + n)$$

Put the value of $\bar{k}^*$ that solves the equation above in

$$(\delta + n)\bar{k}^* = \sigma A_0 f\left(\bar{k}^*\right)$$

and solve for the savings rate σ. The **golden rule** value of σ is

$$\sigma = \frac{(\delta + n)\bar{k}^*}{A_0 f\left(\bar{k}^*\right)}$$

The Golden Rule

Assume Cobb-Douglas production function $f(k_t) = k_t^{\theta}$.

- Show that $\sigma = \theta$.
- Briefly discuss the intuition.

Stochastic Solow Model

Adding a stochastic shock to the standard Solow model

- a technology shock and a shock to the savings rate are essentially identical - they affect the evolution of output
- other variables: discount factor; growth rate of population

Assume technology is stochastic

- Effects of stochastic technology growth:
  - Basis for RBC theory

Several ways of defining the stochastic process for technology.

Stochastic Solow Model

First-order moving average

$$A_t = \psi \bar{A} + (1-\psi) A_{t-1} + \varepsilon_t$$

where $\psi \in (0,1)$ and $\bar{A} \geq 0$. Probability distribution of ε_t bounded from below by $-\psi \bar{A}$

Alternative process

$$A_t = \bar{A} e^{\varepsilon_t}$$

ε_t: normal distribution, mean 0

- A_t: log normal distribution of the form

$$\ln A_t = \ln \bar{A} + \varepsilon_t$$

Stochastic Solow Model

The first-order difference equation that describes the time path of the economy in the model is mechanical, in the sense that the savings rate is a constant, and equation (1) is written simply as

$$k_{t+1} = \frac{(1-\delta)k_t + \sigma \bar{A} e^{\varepsilon_t} f(k_t)}{1+n} \tag{2}$$

Divide both sides of this equation by k_t to get the growth rate $\gamma_t = k_{t+1}/k_t$, on the LHS, rearrange and then take logarithms.

$$\ln\left[\gamma_t - \frac{1-\delta}{1+n}\right] = \ln \frac{\sigma \bar{A}}{1+n} + \ln \frac{f(k_t)}{k_t} + \varepsilon_t \tag{3}$$

Stochastic Solow Model

Assume a Cobb-Douglas production function - equation (3)

$$\ln\left[\gamma_t - \frac{1-\delta}{1+n}\right] = \varphi - (1-\theta)\ln k_t + \varepsilon_t$$

where $\varphi \equiv \ln[\sigma \bar{A}/(1+n)]$.

- Gross growth rate of per worker capital: nonlinear function of the current per worker capital stock and of the shocks.
- Variance of the growth rate of per worker capital depends on the initial level of per worker capital as well as the variance of the shocks.

Log-Linear Version

Given that the model is nonlinear, a simple expression for the variance of per worker capital in terms of the shocks is not easy to find or even to define in general.

Possible to find a linear approximation of the model around a SS

- Study 2nd-order characteristics of the model
  - e.g., size of the technology shock that is required so that the variance of output is similar to that of real economies around their long-term trend
- This part is important for understanding the process of producing a log-linear version of a model

Log-Linear Version

Define a log differences of a variable $\tilde{X}_t$ as

$$\tilde{X} \quad = \quad \ln X_t - \ln \bar{X}$$

X_t is the time t value of the variable
$\bar{X}$ is its value in the stationary state

This definition of the log differences allows us to write

$$X_t \quad = \quad \bar{X} e^{\tilde{X}_t}$$

Log-Linear Version

Rules for first-order approximations (small values of $\tilde{X}_t$ and $\tilde{Y}_t$)

$$
\begin{aligned}
e^{\tilde{X}_t} &\approx 1 + \tilde{X}_t \\
e^{\tilde{X}_t + a\tilde{Y}_t} &\approx 1 + \tilde{X}_t + a\tilde{Y}_t \\
\tilde{X}_t \tilde{Y}_t &\approx 0 \\
E_t\left[ae^{\tilde{X}_{t+1}}\right] &\approx E_t\left[a\tilde{X}_{t+1}\right] + constant
\end{aligned}
$$

Log-Linear Version: Capital

Stochastic, Cobb-Douglas, zero technology growth version of the first-order difference equation

$$(1+n)k_{t+1} = (1-\delta)k_t + \sigma \bar{A} e^{\varepsilon_t} k_t^{\theta}$$

replace k_j by $\bar{k}e^{\tilde{k}_j}$, where $\tilde{k} = \ln k_j - \ln \bar{k}$.

$$(1+n)\bar{k}e^{\tilde{k}_{t+1}} = (1-\delta)\bar{k}e^{\tilde{k}_t} + \sigma \bar{A} e^{\varepsilon_t} \bar{k}^{\theta} e^{\theta \tilde{k}_t}$$

which becomes (rule approximation)

$$(1+n)\bar{k}(1+\tilde{k}_{t+1}) = (1-\delta)\bar{k}(1+\tilde{k}_t) + \sigma \bar{A}(1+\varepsilon_t)\bar{k}^{\theta}(1+\theta\tilde{k}_t)$$

Log-Linear Version: Capital

The first-order difference equation (approximation)

$$(1+n)\bar{k}(1+\tilde{k}_{t+1}) \quad = \quad (1-\delta)\bar{k}(1+\tilde{k}_t)+\sigma\bar{A}(1+\varepsilon_t)\bar{k}^{\theta}(1+\theta\tilde{k}_t)$$

In the nonstochastic stationary state

$$(1+n)\bar{k} \quad = \quad (1-\delta)\bar{k}+\sigma\bar{A}\bar{k}^{\theta}$$

Removing the nonstochastic stationary state terms from the approximation above gives

$$(1+n)\bar{k}\tilde{k}_{t+1} \quad = \quad (1-\delta)\bar{k}\tilde{k}_t+\sigma\bar{A}\varepsilon_t\bar{k}^{\theta}+\sigma\bar{A}\bar{k}^{\theta}\theta\tilde{k}_t++\sigma\bar{A}\bar{k}^{\theta}\theta\varepsilon_t\tilde{k}_t$$

Log-Linear Version: Capital

$$(1+n)\bar{k}\tilde{k}_{t+1} \quad = \quad (1-\delta)\bar{k}\tilde{k}_t + \sigma\bar{A}\varepsilon_t\bar{k}^\theta + \sigma\bar{A}\bar{k}^\theta\theta\tilde{k}_t + +\sigma\bar{A}\bar{k}^\theta\theta\varepsilon_t\tilde{k}_t$$

Since $\varepsilon_t\tilde{k}_t \approx 0$, this becomes

$$(1+n)\bar{k}\tilde{k}_{t+1} \quad = \quad (1-\delta)\bar{k}\tilde{k}_t + \sigma\bar{A}\varepsilon_t\bar{k}^\theta + \sigma\bar{A}\bar{k}^\theta\theta\tilde{k}_t$$

Combining terms, we arrive at the first-order difference equation

$$\tilde{k}_{t+1} \quad = \quad B\tilde{k}_t + C\varepsilon_t \tag{4}$$

Log-Linear Version: Capital

.... the first-order difference equation

$$\tilde{k}_{t+1} = B\tilde{k}_t + C\varepsilon_t$$

where

$$\begin{aligned} B &= \frac{1-\delta}{1+n} + \frac{\theta\sigma\bar{A}\bar{k}^{\theta-1}}{1+n} = \frac{1-\delta}{1+n} + \frac{\theta(\delta+n)}{1+n} \\ &= \frac{1+\theta n - \delta(1-\theta)}{1+n} < 1 \\ C &= \frac{\sigma\bar{A}\bar{k}^{\theta-1}}{1+n} = \frac{\delta+n}{1+n} \end{aligned}$$

Log-Linear Version: Capital

Recursively substituting $\tilde{k}_{t-j}$ for $j = 0, ..., \infty$ in equation (4) results in the approximation

$$\tilde{k}_{t+1} = C \sum_{i=0}^{\infty} B^i \varepsilon_{t-i} \tag{5}$$

We can use expression (5) to calculate the variance of capital around its stationary state $\tilde{k}_{t+1}$ as

$$\begin{aligned} var\left(\tilde{k}_{t+1}\right) &= E\left[\tilde{k}_{t+1}\tilde{k}_{t+1}\right] \\ &= E\left[\left(C\sum_{i=0}^{\infty} B^i \varepsilon_{t-i}\right)\left(C\sum_{i=0}^{\infty} B^i \varepsilon_{t-i}\right)\right] \end{aligned}$$

Log-Linear Version: Capital

If the technology shocks are independent, i.e. $E\left[\varepsilon_t \varepsilon_s\right] = 0$ if $t \neq s$,

$$var\left(\tilde{k}\right) = C^2 \sum_{i=0}^{\infty} B^{2i} var(\varepsilon) = \frac{C^2}{1-B^2} var(\varepsilon)$$

where

$$\frac{C^2}{1-B^2} var(\varepsilon) = \frac{(\delta+n)^2}{(1+n)^2 - \left[1+\theta n - \delta(1-\theta)\right]^2}$$

Assuming that $\delta = 0.1$, $\sigma = 0.2$, $n = 0.02$ and $\theta = 0.36$

- variance of capital around its stationary state is 0.0955 times the variance of the shock to technology.

Log-Linear Version: Output

Variance of technology shock relative to variance of output.

Log-linear version of the production function (found from)

$$y_t = \bar{A}e^{\varepsilon_t}k_t^{\theta}$$

Replace the variables with shocks around stationary states to get

$$\bar{y}e^{\tilde{y}_t} = \bar{A}e^{\varepsilon_t}\bar{k}^{\theta}e^{\theta\tilde{k}_t}$$

which is approximated by

$$\bar{y}\left(1+\tilde{y}_t\right) = \bar{A}\left(1+\varepsilon_t\right)\bar{k}^{\theta}\left(1+\theta\tilde{k}_t\right)$$

Log-Linear Version: Output

$$\bar{y}\left(1+\tilde{y}_t\right) \quad = \quad \bar{A}\left(1+\varepsilon_t\right)\bar{k}^{\theta}\left(1+\theta\tilde{k}_t\right)$$

Remove the stationary state value of output to get

$$\bar{y}\tilde{y}_t \quad = \quad \bar{A}\bar{k}^{\theta}\varepsilon_t + \bar{A}\bar{k}^{\theta}\theta\tilde{k}_t + \bar{A}\bar{k}^{\theta}\varepsilon_t\theta\tilde{k}_t$$

Since $\varepsilon_t\tilde{k}_t \approx 0$ and $\bar{y} = \bar{A}\bar{k}^{\theta}$, this expression becomes

$$\tilde{y}_t \quad = \quad \varepsilon_t + \theta\tilde{k}_t$$

Log-Linear Version: Output

To find the approximate variance of $\tilde{y}_t$ we use the fact that $\tilde{k}_t$ is independent of ε_t (equation 5) and calculate

$$\begin{aligned} var(\tilde{y}_t) &= E\left[\tilde{y}_t \tilde{y}_t\right] \\ &= E\left[\left(\varepsilon_t + \theta \tilde{k}_t\right)\left(\varepsilon_t + \theta \tilde{k}_t\right)\right] \\ &= var(\varepsilon_t) + \theta^2 var(\tilde{k}_t) \end{aligned}$$

For the same parameter values, we find $var(\tilde{y}_t) = 1.0123 var(\varepsilon_t)$.

- This means that for the Solow model, the variance in output is almost exactly equal to the variance in the technology shock.

Log-Linear Version: Output

For the same parameter values, we find $var(\tilde{y}_t) = 1.0123 var(\varepsilon_t)$.

- This means that for the Solow model, the variance in output is almost exactly equal to the variance in the technology shock.

In the model, the shocks show very little persistence.

- With our parameter values, a technology shock in period t accounts for only 1.23 percent of the variance in output in period $t+1$.
- The Solow model does not do a good job of explaining either the variance in output or the persistence of output shocks.

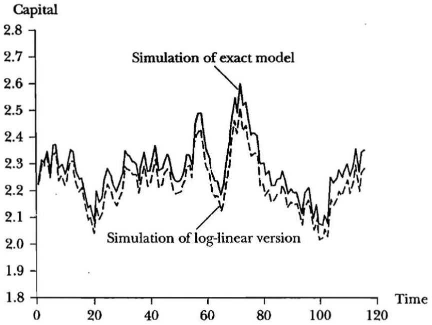


FIGURE 1.3 Simulations of the exact and log-linear Solow model

Figure: Exact vs. Log-Linear Solow Model

The General Solow Model

In the basic Solow model: no growth in GDP per worker in steady state. This contradicts the empirics for the Western world.

In the general Solow model:

- Total factor productivity, B_t, is assumed to grow at a constant, exogenous rate.
  - This implies a steady state with balanced growth and a constant, positive growth rate of GDP per worker.
- The source of long-run growth in GDP per worker in this model is exogenous technological growth.

The Micro World of the General Solow Model

... is the same as in the basic Solow model, except for one difference: the production function.

There is a possibility of technological progress:

$$Y_t = B_t K_t^{\alpha} L_t^{1-\alpha}$$

The full sequence (B_t) is exogenous and $B_t > 0$ for all t.

- Special case is $B_t = B$ (basic Solow model).

The Production Function with Technological Progress

- $Y_t = B_t K_t^{\alpha} L_t^{1-\alpha}$ with a given sequence $(B_t) \iff$
- $Y_t = K_t^{\alpha} (A_t L_t)^{1-\alpha}$ with a given sequence (A_t), where $A_t \equiv (B_t)^{1/1-\alpha}$

With a Cobb-Douglas production function it makes no difference whether we describe technical progress by a sequence, (B_t), for TFP or by the corresponding sequence, (A_t), for labour augmenting productivity. In our case, the latter is the most convenient.

The Production Function with Technological Progress

The exogenous sequence, (A_t), is given by:

$$\begin{aligned} A_{t+1} &= (1+g)A_t, \quad g > -1 \\ A_t &= (1+g)^t A_0, \end{aligned}$$

Technical progress comes as "manna from heaven" (it requires no input of production).

The General Solow Model

Remember the definitions: $y_t \equiv Y_t / L_t$ and $k_t \equiv K_t / L_t$.

Dividing by L_t on both sides of $Y_t = K_t^\alpha \left(A_t L_t\right)^{1-\alpha}$ gives the per capita production function:

$$y_t = k_t^\alpha A_t^{1-\alpha}$$

From this follows:

$$\begin{aligned} \ln y_t - \ln y_{t-1} &= \alpha \left(\ln k_t - \ln k_{t-1}\right) + \left(1-\alpha\right)\left(\ln A_t - \ln A_{t-1}\right) \\ g_t^y &= \alpha g_t^k + \left(1-\alpha\right) g_t^A \\ g_t^y &\cong \alpha g_t^k + \left(1-\alpha\right) g \end{aligned}$$

The General Solow Model

$$g_t^y \cong \alpha g_t^k + (1-\alpha)\, g$$

Growth in y_t can come from exactly two sources

- g_t^y is the weighted average of g_t^k and g with weights α and $(1-\alpha)$.
- If, as in balanced growth, k_t / y_t is constant, then $g_t^y = g$!

The Complete Model

$$
\begin{aligned}
Y_t &= K_t^{\alpha} \left(A_t L_t\right)^{1-\alpha} \\
r_t &= \alpha \left(\frac{K_t}{A_t L_t}\right)^{\alpha-1} \\
w_t &= (1-\alpha) \left(\frac{K_t}{A_t L_t}\right)^{\alpha} A_t \\
S_t &= s Y_t \\
K_{t+1} - K_t &= S_t - \delta K_t \\
L_{t+1} &= (1+n) L_t \\
A_{t+1} &= (1+g) A_t
\end{aligned}
$$

Parameters: α, s, δ, g and n.

State variables: K_t, L_t, and A_t; given K_0, L_0 and A_0.

The Complete Model

Note:

$$\begin{aligned} r_t K_t &= \alpha \left(K_t\right)^{\alpha} \left(A_t L_t\right)^{1-\alpha} = \alpha Y_t \\ w_t L_t &= (1-\alpha) \left(K_t\right)^{\alpha} \left(A_t L_t\right)^{1-\alpha} = (1-\alpha) Y_t \end{aligned}$$

- Capital's share α, labour's share $(1-\alpha)$, pure profits is zero.

Define "effective labour" as $\tilde{L}_t = A_t L_t$:

$$\tilde{L}_{t+1} = (1+n)(1+g)\tilde{L}_t = (1+\tilde{n})\tilde{L}_t$$

The model is mathematically equivalent to the basic Solow model with $\tilde{L}_t$ taking the place of L_t, and $\tilde{n}$ taking the place of n, and with $B = 1$!

Analyzing the General Solow Model

If the model implies convergence to a steady state with balanced growth, then in steady state k_t and y_t must grow at the same constant rate. Also recall:

$$g_t^y = \alpha g_t^k + (1-\alpha)\, g_t^A$$

Hence, if $g_t^y = g_t^k$, then $g_t^y = g_t^k = g_t^A$.

- If there is convergence towards a steady state with balanced growth, then in this steady state k_t and y_t will both grow at the same rate as B_t and hence k_t/A_t and y_t/A_t will be constant.

$K_t/\tilde{L}_t = K_t/(A_t L_t) = k_t/A_t$ and $Y_t/\tilde{L}_t = Y_t/(A_t L_t) = y_t/A_t$ converge towards constant steady state values.

Analyzing the General Solow Model

1. Define: $\tilde{y}_t = y_t/A_t = Y_t/A_tL_t$ and $\tilde{k}_t = k_t/A_t = K_t/A_tL_t$.
2. From $Y_t = K_t^{\alpha}(A_tL_t)^{1-\alpha}$, we get $\tilde{y}_t = \tilde{k}_t^{\alpha}$
3. From $S_t = sY_t$ and $K_{t+1} - K_t = S_t - \delta K_t$ to get:

$$K_{t+1} = sY_t + (1-\delta)K_t$$

4. Divide by $A_{t+1}L_{t+1} = (1+g)(1+n)A_tL_t$ on both sides to find that:

$$\tilde{k}_{t+1} = \frac{1}{(1+n)(1+g)}\left(s\tilde{y}_t + (1-\delta)\tilde{k}_t\right)$$

Analyzing the General Solow Model

5. Insert $\tilde{y}_t = \tilde{k}_t^{\alpha}$ to get the **transition equation**:

$$\tilde{k}_{t+1} = \frac{1}{(1+n)(1+g)}\left[s\tilde{k}_t^{\alpha} + (1-\delta)\tilde{k}_t\right]$$

6. Subtracting $\tilde{k}_t$ from both sides of the transition equation gives the **Solow equation**:

$$\tilde{k}_{t+1} - \tilde{k}_t = \frac{1}{(1+n)(1+g)}\left[s\tilde{k}_t^{\alpha} - (n+g+\delta+ng)\tilde{k}_t\right]$$

Analyzing the General Solow Model

The **transition equation**:

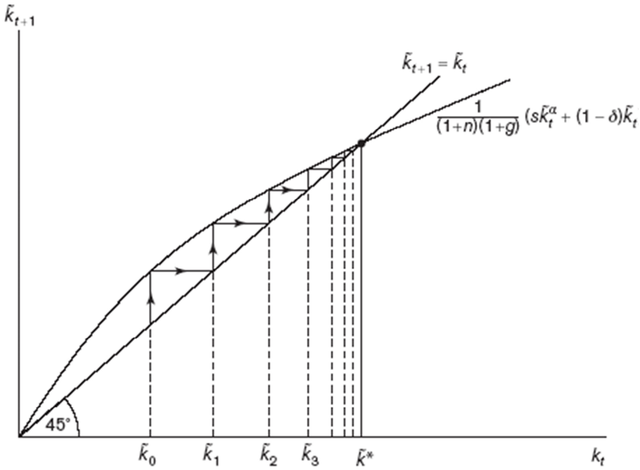
$$\tilde{k}_{t+1} = \frac{1}{(1+n)(1+g)} \left[s\tilde{k}_t^{\alpha} + (1-\delta)\tilde{k}_t \right]$$

The slope of the transition curve at any $\tilde{k}_t$ is:

$$\frac{d\tilde{k}_{t+1}}{d\tilde{k}_t} = \frac{s\alpha\tilde{k}_t^{\alpha-1} + (1-\delta)}{(1+n)(1+g)}$$

$\lim_{\tilde{k}_t \to 0} \frac{d\tilde{k}_{t+1}}{d\tilde{k}_t} = \infty$ and $\lim_{\tilde{k}_t \to \infty} \frac{d\tilde{k}_{t+1}}{d\tilde{k}_t} < 1 \iff (n+g+\delta+ng) > 0.$

Analyzing the General Solow Model



Analyzing the General Solow Model

Some first conclusions:

- In the long run, $\tilde{k}_t$ and $\tilde{y}_t$ converge to constant levels, $\tilde{k}^*$ and $\tilde{y}^*$, respectively. These levels define steady state.
- In steady state, k_t and y_t both grow at the same rate as A_t, that is, at the rate g, and the capital output ratio, $K_t / Y_t = k_t / y_t$ must be constant.

Steady State

The Solow equation:

$$\tilde{k}_{t+1} - \tilde{k}_t = \frac{1}{(1+n)(1+g)} \left[s\tilde{k}_t^{\alpha} - (n+g+\delta+ng)\tilde{k}_t \right]$$

Together with $\tilde{k}_{t+1} = \tilde{k}_t = \tilde{k}^*$ gives:

$$\begin{aligned} \tilde{k}^* &= \left(\frac{s}{n+g+\delta+ng} \right)^{\frac{1}{1-\alpha}} \\ \tilde{y}^* &= \left(\frac{s}{n+g+\delta+ng} \right)^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

Steady State

Using $\tilde{y}_t = y_t / A_t$ and $\tilde{k}_t = k_t / A_t$ we get the steady state growth paths:

$$\begin{aligned} k_t^* &= A_t \left(\frac{s}{n+g+\delta+ng} \right)^{\frac{1}{1-\alpha}} \\ y_t^* &= A_t \left(\frac{s}{n+g+\delta+ng} \right)^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

Since $c_t = (1-s)y_t$, then

$$c_t^* = (1-s)A_t \left(\frac{s}{n+g+\delta+ng} \right)^{\frac{\alpha}{1-\alpha}}$$

Steady State

It also follows from $r_t = \alpha \left(\tilde{k}_t\right)^{\alpha-1}$ and $w_t = (1-\alpha)A_t \left(\tilde{k}_t\right)^{\alpha}$

$$\begin{aligned} r^* &= \alpha \left(\frac{s}{n+g+\delta+ng}\right)^{-1} \\ w^* &= (1-\alpha)A_t \left(\frac{s}{n+g+\delta+ng}\right)^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

There is balanced growth in steady state: k_t, y_t and w_t grow at the same constant rate, g, and r_t is constant.

- There is positive growth in GDP per capita in steady state (provided that $g > 0$).

Steady State

$$
\begin{aligned}
y_t^* &= A_0 (1+g)^t \left(\frac{s}{n+g+\delta+ng} \right)^{\frac{\alpha}{1-\alpha}} \\
c_t^* &= (1-s) A_0 (1+g)^t \left(\frac{s}{n+g+\delta+ng} \right)^{\frac{\alpha}{1-\alpha}}
\end{aligned}
$$

Golden rule: the s that maximizes the entire path, c_t^*.

- Again: $s^{**} = \alpha$.

The elasticities of y_t^* w.r.t. s and $n+g+\delta$ are again $\alpha / (1-\alpha)$ and $-\alpha / (1-\alpha)$, respectively.

General Solow Model: Some Conclusions

Using the transition equation and the transition diagram we showed convergence to steady state: $\tilde{k}^*, \tilde{y}^*$.

We derived some relevant steady state growth paths, e.g.,

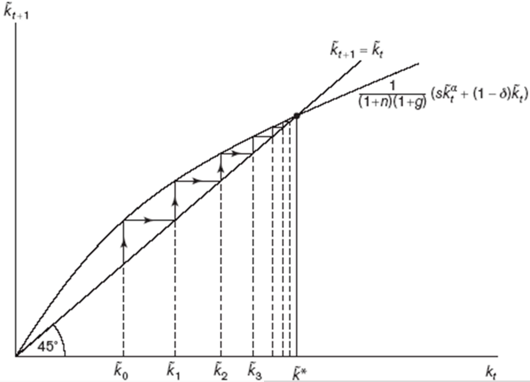
$$y_t^* = A_0(1+g)^t \left(\frac{s}{n+g+\delta+ng} \right)^{\frac{\alpha}{1-\alpha}}$$

We showed that there is balanced growth in steady state with k_t, y_t and w_t growing at the same positive growth rate, g, and with a constant real interest rate, $r^* - \delta$.

We showed and discussed empirics for steady state: The model substantially underestimates the impact of the structural parameters on GDP per worker!

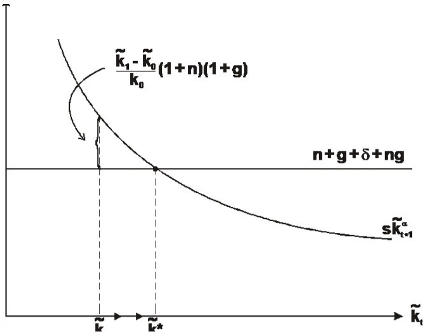
Solow Diagrams

$$\tilde{k}_{t+1} - \tilde{k}_t = \frac{1}{(1+n)(1+g)}\left[s\tilde{k}_t^{\alpha} - (n+g+\delta+ng)\tilde{k}_t\right]$$



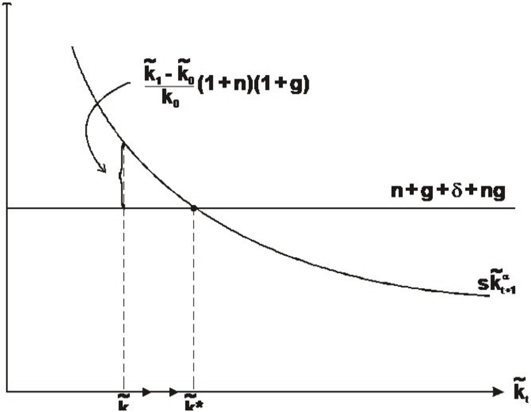
Solow Diagrams

$$\frac{\tilde{k}_{t+1} - \tilde{k}_t}{\tilde{k}_t} = \frac{1}{(1+n)(1+g)} \left[s\tilde{k}_t^{\alpha-1} - (n+g+\delta+ng) \right]$$

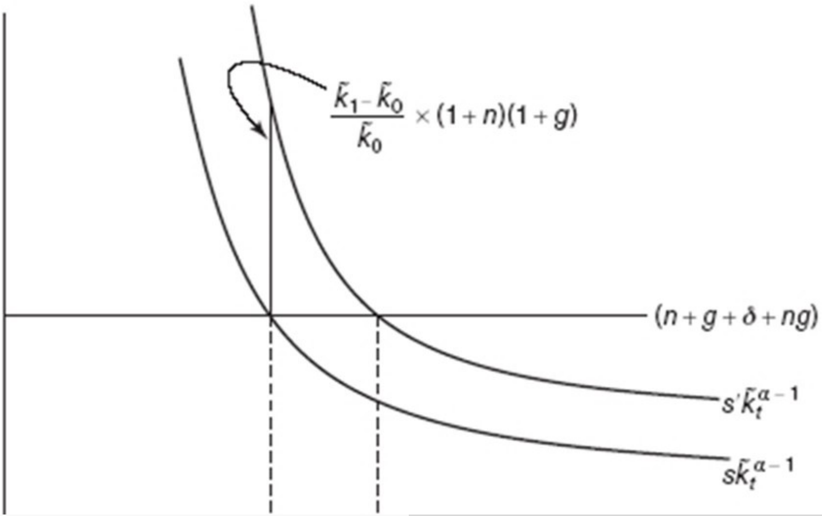


Comparative Analysis in the Solow Diagrams

Initially the economy is in steady state at α, s, δ, n and g.
The savings rate increases permanently from s to $s' > s$.

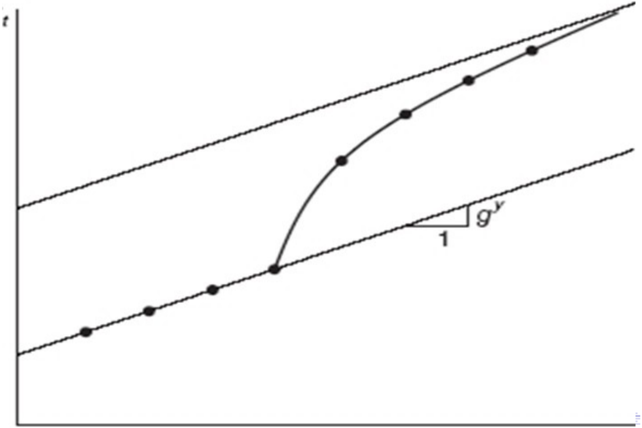


Comparative Analysis in the Solow Diagrams



Comparative Analysis in the Solow Diagrams

Old steady state: $\tilde{k}_t = k_t / A_t$ and $\tilde{y}_t = y_t / A_t$, and k_t and y_t both grow at rate g, the lower line below:



Comparative Analysis in the Solow Diagrams

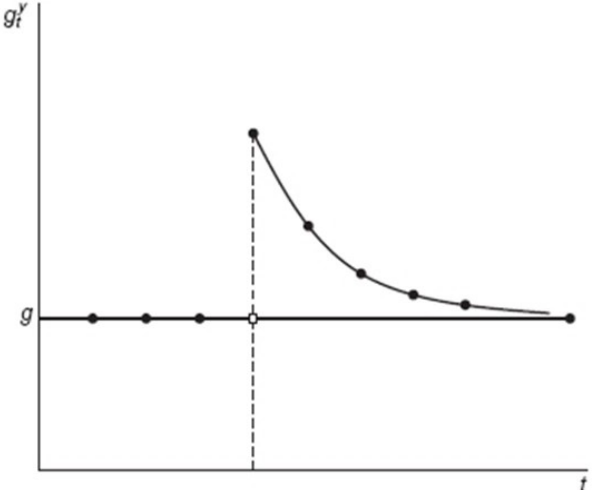
Transition: $\tilde{k}_t = k_t / A_t$ grows from $\tilde{k}^*$ up to $\tilde{k}^{*\prime}$.

- The growth rate of $\tilde{k}_t$ jumps up and then gradually falls back to zero.
- From $k_t = A_t \tilde{k}_t$ follows that $g_t^k = g_t^{\tilde{k}} + g_t^A$.

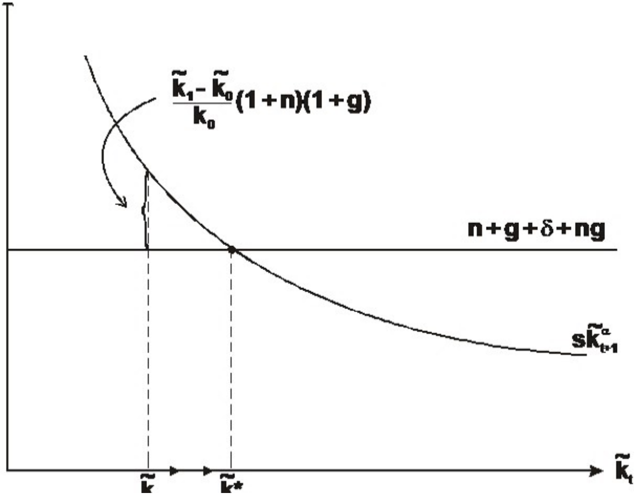
During the transition, k_t grows at a larger rate than g, and g_t^k jumps up and then falls gradually back to g.

The growth rate of y_t jumps too, since $g_t^y = \alpha g_t^k + (1 - \alpha)\, g^A$.

Comparative Analysis in the Solow Diagrams



Convergence in the Solow Model



The General Solow Model: Conclusions

- Implications for economic policies are more or less the same as those derived from the basic Solow model.
- The model implies convergence to a steady state with balanced growth and with a constant, positive growth rate of GDP per worker. Thus, the steady state prediction of the model is in accordance with a fundamental "stylized fact". However, the underlying source of growth, technological progress, is not explained.
- The steady state prediction performs quite well empirically, but the model underestimates the effect of the savings rate and the growth rate of the labour force on income per worker.
- The convergence prediction also performs well empirically, but the model overestimates the rate of convergence.

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