Real Business Cycles Theory and Dynare

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Some Facts about Economic Fluctuations

Modern economies undergo significant short-run variations in aggregate output and employment.

U.S. economy: severe contraction in 2007 - 2009.

- From the fourth quarter of 2007 to the second quarter of 2009, real GDP fell 3.8 percent.
- Fraction of the adult population employed fell by 3.1 percentage points
- Unemployment rate rose from 4.8 to 9.3 percent.

Some Facts about Economic Fluctuations

- Over the previous 5 years (fourth quarter of 2002-2007),
  - *Real GDP rose at an average annual rate of 2.9 percent*
  - The fraction of the adult population employed rose by 0.3% points.
  - *Unemployment rate fell from 5.9 to 4.8 percent.*

- *Understanding the causes of aggregate fluctuations is a central goal of macroeconomics.*

Some Facts about Economic Fluctuations

1. Fluctuations do not exhibit any simple regular or cyclical pattern.

- Figure 5.1 plots *seasonally adjusted real GDP* per person since 1947

Output declines vary considerably in size and spacing.

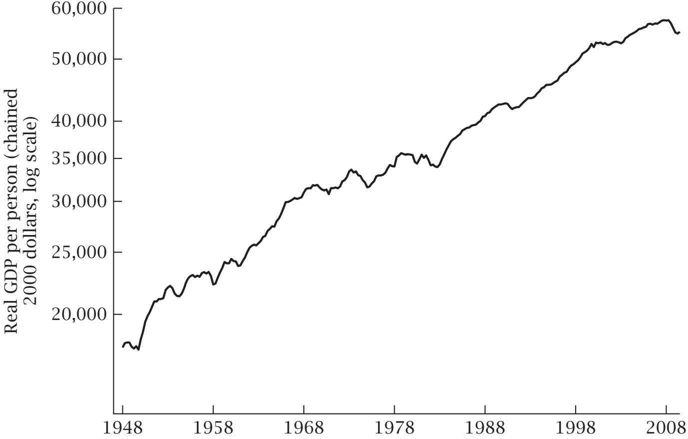


FIGURE 5.1 U.S. real GDP per person, 1947:1–2009:3

Some Facts about Economic Fluctuations

2. Fluctuations are distributed very unevenly over the components of output.

Inventory investment

- On average accounts for only a trivial fraction of GDP

Its fluctuations account for close to half of the shortfall in growth relative to normal in recessions:

- Inventory accumulation is on average large and positive at peaks, and large and negative at troughs.

Some Facts about Economic Fluctuations

2. Fluctuations are distributed very unevenly over the components of output.

- *Consumer purchases of durable goods, residential investment* (that is, housing), and *fixed nonresidential investment* (that is, business investment other than inventories) also account for disproportionate shares of output fluctuations.
- Consumer purchases of nondurables and services, government purchases, and net exports are relatively stable.

** **Output components that decline disproportionately when aggregate output is falling also rise disproportionately when output is growing at above-normal rates.**

Some Facts about Economic Fluctuations

3. Asymmetries in output movements.

There are no large asymmetries between rises and falls in output

- Output growth is distributed roughly symmetrically around its mean.
- Output seems to be characterized by relatively long periods when it is slightly above its usual path, interrupted by brief periods when it is relatively far below.

Changes in the magnitude of fluctuations over time(*)

- Macroeconomic history of the United States since the late 1800s

1. Period before the Great Depression;
2. the Depression and World War II;
3. the period from the end of World War II to about the mid-1980s;
4. the mid-1980s to the present.

- Fluctuations before the Depression were only moderately larger than in the period from World War II to the mid-1980s.
- Output movements in the era before the Depression appear slightly larger, and slightly less persistent, than in the period following World War II;
- No sharp change in the character of fluctuations.

Changes in the magnitude of fluctuations over time(*)

The remaining two periods are the extremes.

- The collapse of the economy in the Depression and the rebound of the 1930s and World War II dwarf any fluctuations before or since.
  - Real GDP in the United States fell by 27 percent between 1929 and 1933, with estimated unemployment reaching 25 percent in 1933.
- Over the next 11 years, real GDP rose at an average annual rate of 10 percent; as a result, unemployment in 1944 was 1.2 percent.
- Finally, real GDP declined by 13 percent between 1944 and 1947, and unemployment rose to 3.9 percent.

Changes in the magnitude of fluctuations over time(*)

In contrast, the period following the recovery from the 1981–1982 recession was one of unprecedented macroeconomic stability

- This period has come to be known as the **"Great Moderation."**
- From 1982 to 2007, the United States underwent only two mild recessions, separated by the longest expansion on record.
- The crisis that began in 2007 represents a sharp change from the economic stability of recent decades.

But one severe recession is not enough to bring average volatility since the mid-1980s even close to its average in the early postwar decades.

Changes in the magnitude of fluctuations over time

1. **Employment falls and unemployment rises during recessions.**
2. **The length of the average workweek falls.**
    - *The declines in employment and in hours in the economy* (though not in the manufacturing sector) *are generally small relative to the falls in output.*
3. **Productivity almost always declines during recessions.**
    - *Declines in productivity and hours imply that the movements in the unemployment rate are smaller than the movements in output.*
    - Okun's law: relationship between changes in output and unemployment rate: $u_t - u_{t-1} = \theta(g_{yt} - g_{\bar{y}})$
4. **Inflation shows no clear pattern.**
    - Nominal and real interest rates generally decline, while the real money stock shows no clear pattern.
    - The real wage tends to fall slightly in recessions.

Overview of Business-Cycle Research

** **Can aggregate fluctuations be understood using a Walrasian model?**

- That is, a competitive model without any externalities, asymmetric information, missing markets, or other imperfections.

The **Ramsey model** is the natural Walrasian baseline model of the aggregate economy

- The model excludes not only market imperfections, but also all issues raised by heterogeneity among households.

We extend the **Ramsey model to incorporate aggregate fluctuations**.

Overview of Business-Cycle Research

1. There must be a source of disturbances

- Without shocks, a Ramsey economy converges to a balanced growth path and then grows smoothly.

- **Technology shocks** change the amount that is produced from a given quantity of inputs
- **Government-purchases shocks** change the quantity of goods available to the private economy for a given level of production.
  - Models are known as *real-business-cycle (or RBC) models*.

Overview of Business-Cycle Research

2. Ramsey model to allow for variations in employment.

- **RBC theory**: question of whether a Walrasian model provides a good description of the main features of observed fluctuations.

- Models in this literature therefore allow for *changes in employment by making households' utility depend not just on their consumption but also on the amount they work*
  - Employment is then determined by labor supply and labor demand.
  - RBC models do a poor job of explaining actual fluctuations.

Overview of Business-Cycle Research

Ultimate goal of business-cycle research

- Building a general-equilibrium model from microeconomic foundations and a specification of the underlying shocks that explains, both qualitatively and quantitatively, the main features of macroeconomic fluctuations.

Fully specified general-equilibrium models of fluctuations are known as dynamic stochastic general-equilibrium (or DSGE) models.

When they are quantitative and use additional evidence to choose parameter values and properties of the shocks, they are calibrated DSGE models.

Overview of Business-Cycle Research

One way that the RBC models appear to fail involves the **effects of monetary disturbances.**

There is strong evidence that contrary to the predictions of the models, such disturbances have important real effects.

As a result, there is broad (though not universal) agreement that nominal imperfections or rigidities are important to macroeconomic fluctuations.

Baseline Real-Business-Cycle Model

Goal: describe the quantitative behavior of the economy

- Specific functional forms for the production and utility functions.

The economy consists of a large number of identical, price-taking firms and a large number of identical, price-taking households.

- Households are infinitely lived.
- Inputs to production: capital (K), labor (L), and "technology" (A).

Output in period t is

$$Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}, \quad 0 < \alpha < 1. \tag{1}$$

Baseline Real-Business-Cycle Model

Output is divided among consumption (C), investment (I), and government purchases (G).

Capital stock in period $t+1$ is

$$K_{t+1} = K_t + I_t - \delta K_t = K_t + Y_t - C_t - G_t - \delta K_t \tag{2}$$

- Fraction δ of capital depreciates each period.

The **government's purchases** are financed by lump-sum taxes that are assumed to equal the purchases each period.

Baseline Real-Business-Cycle Model

Labor and capital are paid their marginal products.

Real wage and the **real interest rate** in period t are

$$\begin{aligned} w_t &= (1-\alpha)K_t^{\alpha}(A_tL_t)^{-\alpha}A_t \\ &= (1-\alpha)\left(\frac{K_t}{A_tL_t}\right)^{\alpha}A_t \end{aligned} \tag{3}$$

$$r_t = \alpha\left(\frac{A_tL_t}{K_t}\right)^{1-\alpha} - \delta \tag{4}$$

Baseline Real-Business-Cycle Model

The representative household maximizes the expected value of

$$U = \sum_{t=0}^{\infty} e^{-\rho t} u(c_t, 1 - l_t) \frac{N_t}{H} \tag{5}$$

$u(\cdot)$ is the instantaneous utility function of the representative member of the household, and ρ is the discount rate.

N_t is **population** and H is the **number of households**

- $\frac{N_t}{H}$ is the number of members of the household.

Baseline Real-Business-Cycle Model

Population grows exogenously at rate n:

$$\ln N_t = \bar{N} + nt, \quad n < \rho. \tag{6}$$

The level of N_t is given by: $N_t = e^{\bar{N}+nt}$.

Since all households are the same, $c = C/N$ and $l = L/N$.

Utility is log-linear:

$$u_t = \ln c_t + b\ln(1 - l_t), \quad b > 0. \tag{7}$$

Baseline Real-Business-Cycle Model

In the absence of any shocks, $\ln A_t$ would be $A + gt$, where g is the rate of technological progress.....

But technology is also subject to random disturbances.

$$\ln A_t = A + gt + \tilde{A}_t, \tag{8}$$

where $\tilde{A}$ reflects departures from trend and follows a first-order autoregressive process.

$$\tilde{A}_t = \rho_A \tilde{A}_{t-1} + \varepsilon_{A,t}, \qquad -1 < \rho_A < 1 \tag{9}$$

where the $\varepsilon_{A,t}$'s are white-noise disturbances (a series of mean-zero shocks that are uncorrelated with one another)

Baseline Real-Business-Cycle Model

$$\tilde{A}_t = \rho_A \tilde{A}_{t-1} + \varepsilon_{A,t}, \qquad -1 < \rho_A < 1$$

- Equation (9) states that the random component of $\ln A_t$, $\tilde{A}_t$, equals fraction ρ_A of the previous period's value plus a random term.
- If ρ_A is positive, this means that the effects of a shock to technology disappear gradually over time.

We make similar assumptions about government purchases G_t.

Household Behavior

The two most important differences between this model and the Ramsey model are the

1. inclusion of leisure in the utility function and
2. introduction of randomness in technology and government purchases.

Discuss the implications of these features for households' behavior.

Intertemporal Substitution in Labor Supply

- Household lives only for *one period* and has no initial wealth.
- Household has only one member.

The Lagrangian for the household's maximization problem is

$$\mathscr{L} = \ln c + b \ln(1 - l) + \lambda(wl - c) \tag{10}$$

The first-order conditions for c and l, respectively, are

$$\frac{1}{c} - \lambda = 0 \tag{11}$$

$$-\frac{b}{1-l} + \lambda w = 0 \tag{12}$$

Intertemporal Substitution in Labor Supply

Substituting (11) and $c = wl$ into (12) yields

$$-\frac{b}{1-l} + \frac{1}{l} = 0 \tag{13}$$

The wage does not enter (13).

- **Labor supply is independent of the wage.**
    - Intuitively, because utility is logarithmic in consumption and the household has no initial wealth, the income and substitution effects of a change in the wage offset each other.

The fact that the level of the wage does not affect labor supply in the static case does not mean that variations in the wage do not affect labor supply when the household's horizon is more than one period.

Intertemporal Substitution in Labor Supply

Household lives for *two periods*.

$$\begin{aligned} \mathscr{L} &= \ln c_1 + b\ln(1-l_1) + e^{-\rho}\left[\ln c_2 + b\ln(1-l_2)\right] \\ &\quad +\lambda\left(w_1 l_1 + \frac{1}{1+r}w_2 l_2 - c_1 - \frac{1}{1+r}c_2\right) \end{aligned} \tag{14}$$

- Equilibrium condition:

$$\frac{1-l_1}{1-l_2} = \frac{1}{e^{-\rho}(1+r)}\frac{w_2}{w_1} \tag{15}$$

Relative labor supply in the two periods responds to the relative wage.

- If, for example, w_1 rises relative to w_2, the household decreases first-period leisure relative to second-period leisure (that is, it increases first-period labor supply relative to second-period supply).

Intertemporal Substitution in Labor Supply

Logarithmic utility: elasticity of substitution between leisure in the two periods is 1.

Equation (15) also implies that **a rise in r raises first-period labor supply relative to second-period supply**.

$$\frac{1-l_1}{1-l_2} = \frac{1}{e^{-\rho}(1+r)} \frac{w_2}{w_1}$$

- Intuitively, a rise in r increases the attractiveness of working today and saving relative to working tomorrow.
  - This effect of the interest rate on labor supply is crucial to employment fluctuations in real-business-cycle models.
- These responses of labor supply to the relative wage and the interest rate are known as **intertemporal substitution in labor supply**

Household Optimization under Uncertainty

The second way that the household's optimization problem differs from its problem in the Ramsey model is that it faces **uncertainty about rates of return and future wages.**

- Because of this uncertainty, the household does not choose deterministic paths for consumption and labor supply.

Households' choices of c and l at any date potentially depend on all the shocks to technology and government purchases up to that date.

Household Optimization under Uncertainty

A complete description of the household's behavior is quite complicated.

- We can describe key features of its behavior without fully solving its optimization problem.

In the Ramsey model, we were able to derive an equation relating present consumption to the interest rate and consumption a short time later (the Euler equation) before imposing the budget constraint and determining the level of consumption.

- With uncertainty, the analogous equation relates consumption in the current period to expectations concerning interest rates and consumption in the next period.

Simplifying Assumptions

The model presented so far cannot be solved analytically.

- Mixture of ingredients that are linear (such as depreciation and the division of output into consumption, investment, and government purchases) and ones that are log-linear (such as the production function and preferences)

A simplified version of the model.

- Eliminate government and 100 percent depreciation each period.
- Equations (2) and (4) become

$$K_{t+1} = Y_t - C_t \tag{16}$$

$$1 + r_t = \alpha \left(\frac{A_t L_t}{K_t} \right)^{1-\alpha} \tag{17}$$

Solving the Model

Because markets are competitive, externalities are absent, and there are a finite number of individuals, the model's equilibrium must correspond to the Pareto optimum.

- We can find the equilibrium either by ignoring markets and finding the social optimum directly, or by solving for the competitive equilibrium. We will take the second approach.

There are **two state variables** in the model:

1. the **capital stock** inherited from the previous period
2. the current value of **technology**.
    - The economy's situation in a given period is described by these two variables.

Solving the Model

The two **endogenous** variables are **consumption** and **employment**.

- Because the endogenous variables are growing over time, it is easier to focus on the fraction of output that is saved, s, and labor supply per person, l.

Our basic strategy:

- Rewrite the equations of the model in log-linear form, substituting $(1-s)Y$ for C whenever it appears.
- Determine how l and s must depend on the current technology and on the capital stock inherited from the previous period to satisfy the equilibrium conditions.

Solving the Model

We will focus on the two conditions for household optimization

$$\frac{1}{c_t} = e^{-\rho} E_t \left[\frac{1}{c_{t+1}} (1 + r_{t+1}) \right] \tag{18}$$

$$\frac{c_t}{1 - l_t} = \left[\frac{w_t}{b} \right] \tag{19}$$

- Intuitively, the combination of logarithmic utility, Cobb–Douglas production, and 100 percent depreciation causes movements in both technology and capital to have offsetting income and substitution effects on saving.

- *It is the fact that* **s** *is constant that allows the model to be solved analytically.*

Solving the Model

Since $c_t = (1 - s_t) Y_t / N_t$, rewriting (18)

$$-\ln\left[(1-s)\frac{Y_t}{N_t}\right] = -\rho + \ln E_t\left[\frac{1+r_{t+1}}{(1-s_{t+1})Y_{t+1}/N_{t+1}}\right] \tag{20}$$

- Equation (17) implies that $1 + r_{t+1} = \alpha\left(\frac{Y_{t+1}}{K_{t+1}}\right)$
- 100 percent depreciation implies: $K_{t+1} = Y_t - C_t = s_t Y_t$.

Substituting these facts into (20) yields

$$\begin{aligned} -\ln(1-s_t) - \ln Y_t + \ln N_t &= -\rho + \ln\alpha + \ln N_t + n \\ &\quad - \ln s_t - \ln Y_t + \ln E_t\left[\frac{1}{(1-s_{t+1})}\right] \end{aligned} \tag{21}$$

Solving the Model

Equation (21) simplifies to

$$\ln s_t - \ln(1 - s_t) = -\rho + \ln \alpha + n + \ln E_t \left[\frac{1}{(1 - s_{t+1})} \right] \tag{22}$$

- Crucially, the two state variables, A and K, do not enter (22).
- There is a constant value of s that satisfies this condition.

Solving the Model

- If s is constant at some value $\hat{s}$, then s_{t+1} is not uncertain, and so $E_t[1/(1-s_{t+1})]$ is simply $1/(1-\hat{s})$.

Thus (22) becomes

$$\ln \hat{s} = \ln \alpha + n - \rho, \tag{23}$$

$$\hat{s} = \alpha e^{n-\rho}. \tag{24}$$

The model has a solution where the saving rate is constant.

Solving the Model

Now consider (19):

$$\frac{c_t}{1 - l_t} = \left[\frac{w_t}{b}\right]$$

Since $c_t = C_t / N_t = (1 - \hat{s}) Y_t / N_t$, we can rewrite this condition as

$$\ln\left[(1 - \hat{s})\frac{Y_t}{N_t}\right] - \ln(1 - l_t) = \ln w_t - \ln b. \tag{25}$$

Since the production function is Cobb–Douglas, $w_t = (1 - \alpha) Y_t / (l_t N_t)$. Substituting this fact into (25) yields.....

Solving the Model

$$\begin{aligned} & \ln(1-\hat{s}) + \ln Y_t - \ln N_t - \ln(1-l_t) \\ = \; & \ln(1-\alpha) + \ln Y_t - \ln l_t - \ln N_t - \ln b \end{aligned} \tag{26}$$

Straightforward algebra yields

$$l_t = \frac{1-\alpha}{(1-\alpha) + b(1-\hat{s})} = \hat{l} \tag{27}$$

Solving the Model

$$l_t = \frac{1-\alpha}{(1-\alpha)+b(1-\hat{s})} = \hat{l}$$

Labor supply is also constant.

- The reason this occurs despite households' willingness to substitute their labor supply intertemporally is that movements in either technology or capital have offsetting impacts on the relative-wage and interest-rate effects on labor supply.
  - An improvement in technology, for example, raises current wages relative to expected future wages, and thus acts to raise labor supply. But, by raising the amount saved, it also lowers the expected interest rate, which acts to reduce labor supply. In the specific case we are considering, these two effects exactly balance.

Solving the Model

The remaining equations of the model do not involve optimization

- They follow from technology, accounting, and competition.

We have found a solution to the model with s and l constant.

Any competitive equilibrium of this model is also a solution to the problem of maximizing the expected utility of the representative household.

- Standard results about optimization imply that this problem has a unique solution (see Stokey, Lucas, and Prescott, 1989).
    - The equilibrium we have found must be the only one.

Discussion

This model: an example of an economy where real shocks drive output movements.

- Because the economy is Walrasian, the movements are the optimal responses to the shocks.
- Contrary to the conventional wisdom about macroeconomic fluctuations, here fluctuations do not reflect any market failures, and government interventions to mitigate them can only reduce welfare.

The implication of real-business-cycle models, in their strongest form, is that observed aggregate output movements represent the time-varying Pareto optimum.

Discussion

The specific form of the output fluctuations implied by the model is determined by the dynamics of technology and the behavior of the capital stock.

The production function, equation (1) implies

$$\ln Y_t = \alpha \ln K_t + (1-\alpha)(\ln A_t + \ln L_t) \tag{28}$$

We know that $K_t = \hat{s} Y_{t-1}$ and $L_t = \hat{l} N_t$; and using (6) and (8)

$$\begin{aligned} \ln Y_t \quad = \quad & \alpha \ln \hat{s} + \alpha \ln Y_{t-1} + (1-\alpha)(\bar{A} + gt) \\ & + (1-\alpha)\tilde{A}_t + (1-\alpha)(\ln \hat{l} + \bar{N} + nt) \end{aligned} \tag{29}$$

Discussion

The two components of the right-hand side of (29) that do not follow deterministic paths are $\alpha \ln Y_{t-1}$ and $(1-\alpha)\tilde{A}_t$.

It must therefore be possible to rewrite (29) in the form

$$\tilde{Y}_t = \alpha \tilde{Y}_{t-1} + (1-\alpha)\tilde{A}_t \tag{30}$$

where $\tilde{Y}_t$ is the difference between $\ln Y_t$ and the value it would take if $\ln A_t$ is equal to $(\bar{A} + gt)$ each period.

Discussion

What does (30) imply concerning the dynamics of output?

Since it holds each period, it implies $\tilde{Y}_{t-1} = \alpha \tilde{Y}_{t-2} + (1-\alpha)\tilde{A}_{t-1}$, or

$$\tilde{A}_{t-1} = \frac{1}{(1-\alpha)}\left(\tilde{Y}_{t-1} - \alpha \tilde{Y}_{t-2}\right) \tag{31}$$

Substituting (9) and (31) into (30), we obtain

$$\tilde{Y}_t = (\alpha + \rho_A)\tilde{Y}_{t-1} - \alpha\rho_A \tilde{Y}_{t-2} + (1-\alpha)\varepsilon_{A,t} \tag{32}$$

Discussion

$$\tilde{Y}_t = (\alpha + \rho_A)\tilde{Y}_{t-1} - \alpha\rho_A\tilde{Y}_{t-2} + (1-\alpha)\varepsilon_{A,t}$$

- Departures of log output from its normal path follow a *second-order autoregressive process*; that is, $\tilde{Y}_t$ can be written as a linear combination of its two previous values plus a white-noise disturbance.
- The combination of a positive coefficient on the first lag of $\tilde{Y}_t$ and a negative coefficient on the second lag can cause output to have a "hump-shaped" response to disturbances.

Discussion

Suppose, for example, that $\alpha = 1/3$ and $\rho_A = 0.9$.

Consider a one-time shock of $1/(1-\alpha)$ to ε_A.

- Using (32) iteratively shows that the shock raises log output relative to the path it would have otherwise followed by 1 in the period of the shock ($1-\alpha$ times the shock), 1.23 in the next period ($\alpha + \rho_A$ times 1), 1.22 in the following period ($\alpha + \rho_A$ times 1.23, minus α times ρ_A times 1), then 1.14, 1.03, 0.94, 0.84, 0.76, 0.68, . . . in subsequent periods.

Because α is not large, the dynamics of output are determined largely by the persistence of the technology shocks, ρ_A.

Discussion

If $\rho_A = 0$ and $\alpha = 1/3$, this implies that almost ninetenths of the initial effect of a shock disappears after only two periods.

- Even if $\rho_A = 1/2$, two-thirds of the initial effect is gone after three periods.

Thus the model does not have any mechanism that translates transitory technology disturbances into significant long-lasting output movements.

These results show that this model yields interesting output dynamics.

- But model does not match major features of fluctuations very well.
    - The saving rate is constant - so that consumption and investment are equally volatile - and labor input does not vary.

Discussion

- The model must be modified if it is to capture many of the major features of observed output movements.

- Introducing depreciation of less than 100 percent and shocks to government purchases improves the model's predictions concerning movements in employment, saving, and the real wage.

Discussion

1. **How lower depreciation improves the fit of the model**
   - Extreme case of no depreciation and no growth, so that investment is zero in the absence of shocks.
   - A positive technology shock, by raising the marginal product of capital in the next period, makes it optimal for households to undertake some investment.
   - Thus the saving rate rises. The fact that saving is temporarily high means that expected consumption growth is higher than it would be with a constant saving rate;
   - From consumers' intertemporal optimization condition this requires the expected interest rate to be higher.
   - But we know: higher interest rate increases current labor supply.

Thus introducing incomplete depreciation causes investment and employment to respond more to shocks.

Discussion

2. Introducing shocks to government purchases

- It breaks the tight link between output and the real wage.
- Since an increase in government purchases increases households' lifetime tax liability, it reduces their lifetime wealth.
- Households consume less leisure - that is, to work more.
- When labor supply rises without any change in technology, the real wage falls;
- thus output and the real wage move in opposite directions.

With shocks to both government purchases and technology, the model can generate an overall pattern of real wage movements that is not strongly procyclical.

Log Linearization techniques

The full model presented cannot be solved analytically.

- Solving models with substantial nonlinearity is often difficult

When the model is relatively simple - find an approximation to the policy function by recursively solving for the value function.

- The problem is to **covert a nonlinear model into a sufficiently good linear approximation**
  - Solutions to the linear approximation are helpful in understanding the behavior of the underlying nonlinear system.

Log Linearization techniques

- A common way of dealing with this problem is to **log-linearize the model around its steady state**.

- That is, **agents' decision rules** and the **equations of motion for the state variables** are replaced by **first-order Taylor approximations in the logs of the relevant variables around the path the economy would follow in the absence of shocks.**

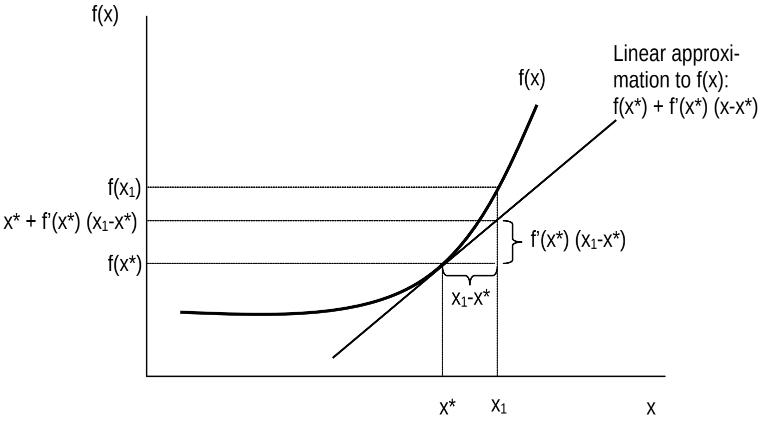
- The assumption is that if the model is not too far from the stationary state, the linear version that results closely approximates the original model.

Log Linearization techniques

- Taylor's Theorem allows us to approximate a function by a straight line around a point at which the function's value is known.

- The basic idea of the **Taylor series**:
  - a general function $f(x)$ can be approximated near a specified value x^* by
$$f(x) \cong f(x^*) + f'(x^*)(x - x^*)$$

- At the specified value x^*
  - value of the function $f(x^*)$ and its first derivative $f'(x^*)$ are known or can be readily calculated.


f(x)
Linear approxi-
mation to f(x):
f(x*) + f'(x*) (x-x*)
f(x)
f(x₁)
x* + f'(x*) (x₁-x*)
f'(x*) (x₁-x*)
f(x*)
x₁-x*
x*
x₁
x

Log Linearization techniques

The straight line is the linear approximation to the nonlinear function $f(x)$ around the value x^*.

- The first-order (linear) Taylor approximation is just the line that is tangent to the function at x^*.
- This approximation will be a good one if x is very close to x^* or if $f(x)$ is nearly linear.

Log Linearization techniques

Consider a nonlinear model that can be represented by a set of equations of the general form

$$F(x_t) = \frac{G(x_t)}{H(x_t)}$$

x_t: vector of the variable of the model that can include expectational variables and lagged variables in addition to contemporaneous variables

The process of **log linearization** is to first take the logarithms of the functions $F()$, $G()$ and $H()$, and then take a first-order Taylor series approximation.

Taking the logarithms gives

$$\ln F(x_t) = \ln G(x_t) - \ln H(x_t)$$

Log Linearization techniques

Taking the **first-order Taylor series expansion around the steady state values**, $\bar{x}$, gives

$$\begin{aligned}\ln F(\bar{x}) + \frac{F'(\bar{x})}{F(\bar{x})}(x_t - \bar{x}) \quad &\approx \quad \ln G(\bar{x}) + \frac{G'(\bar{x})}{G(\bar{x})}(x_t - \bar{x}) \\ &\quad - \ln H(\bar{x}) - \frac{H'(\bar{x})}{H(\bar{x})}(x_t - \bar{x})\end{aligned}$$

$X'(\bar{x})$ indicates the gradient at the stationary state.

- Notice that the **model is now linear in** x_t, since $\frac{F'(\bar{x})}{F(\bar{x})}$, $\frac{G'(\bar{x})}{G(\bar{x})}$, $\frac{H'(\bar{x})}{H(\bar{x})}$, $\ln F(\bar{x})$, $\ln G(\bar{x})$ and $\ln H(\bar{x})$ are constants.

Log Linearization techniques

Given that the log version of the model holds at the stationary state

$$\ln F(\bar{x}) = \ln G(\bar{x}) - \ln H(\bar{x}),$$

eliminate the three $\ln(\cdot)$ components, and the equation simplifies to

$$\frac{F'(\bar{x})}{F(\bar{x})}(x_t - \bar{x}) \approx \frac{G'(\bar{x})}{G(\bar{x})}(x_t - \bar{x}) - \frac{H'(\bar{x})}{H(\bar{x})}(x_t - \bar{x})$$

* The implicit assumption is that one is staying close enough to the stationary state $\bar{x}$ so that the second-order or higher terms of the Taylor expansion are small enough to be irrelevant.

Example: Cobb-Douglas production function

$$Y_t = \lambda_t K_t^{\alpha} {L_t}^{1-\alpha}$$

Take the logarithms of both sides of the production function to get

$$\ln Y_t = \ln \lambda_t + \alpha \ln K_t + (1-\alpha) \ln L_t,$$

Then the first-order Taylor expansion gives

$$\begin{aligned} \ln \bar{Y} + \frac{1}{\bar{Y}}(Y_t - \bar{Y}) \quad &\approx \quad \ln \bar{\lambda} + \frac{1}{\bar{\lambda}}(\lambda_t - \bar{\lambda}) + \alpha \ln \bar{K} + \frac{\alpha}{\bar{K}}(K_t - \bar{K}) \\ &\quad + (1-\alpha) \ln \bar{L} + \frac{(1-\alpha)}{\bar{L}}(L_t - \bar{L}) \end{aligned}$$

Example: Cobb-Douglas production function

Since in a stationary state

$$\ln \bar{Y} = \ln \bar{\lambda} + \alpha \ln \bar{K} + (1-\alpha) \ln \bar{L},$$

the zero-order terms can be removed to get

$$\frac{1}{\bar{Y}}(Y_t - \bar{Y}) \approx \frac{1}{\bar{\lambda}}(\lambda_t - \bar{\lambda}) + \frac{\alpha}{\bar{K}}(K_t - \bar{K}) + \frac{(1-\alpha)}{\bar{L}}(L_t - \bar{L})$$

Further simplification gives

$$\frac{Y_t}{\bar{Y}} + 1 \approx \frac{\lambda_t}{\bar{\lambda}} + \frac{\alpha K_t}{\bar{K}} + \frac{(1-\alpha) L_t}{\bar{L}}$$

The production function is now expressed as a linear equation.

Consider the following figures:

- The effects of a 1 percent technology shock on the paths of technology, capital, labor, consumption, ...
- The effects of a 1 percent government purchases shock on the paths of output, consumption, wage, and interest rate.

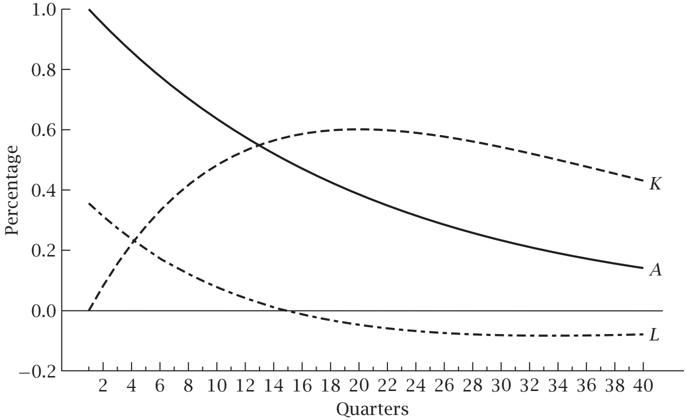


FIGURE 5.2 The effects of a 1 percent technology shock on the paths of technology, capital, and labor

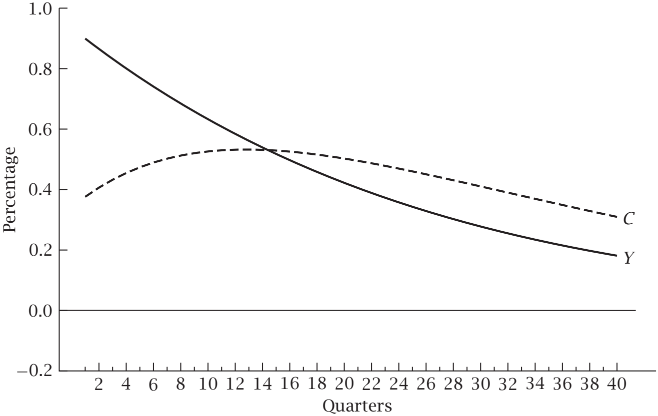


FIGURE 5.3 The effects of a 1 percent technology shock on the paths of output and consumption

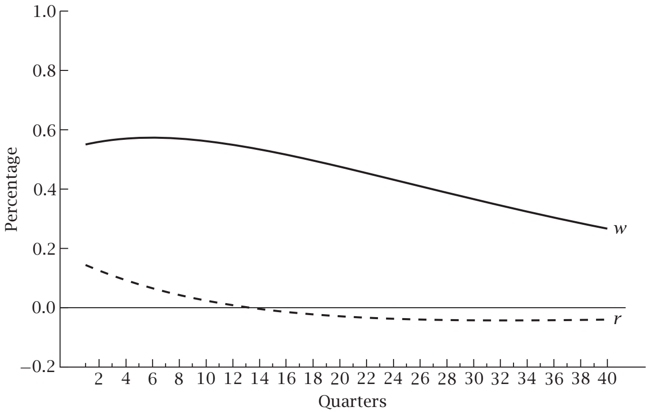


FIGURE 5.4 The effects of a 1 percent technology shock on the paths of the wage and the interest rate

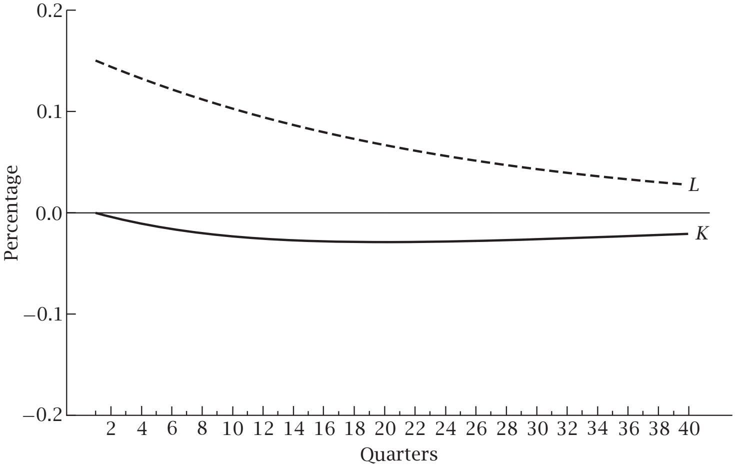


FIGURE 5.5 The effects of a 1 percent government-purchases shock on the paths of capital and labor

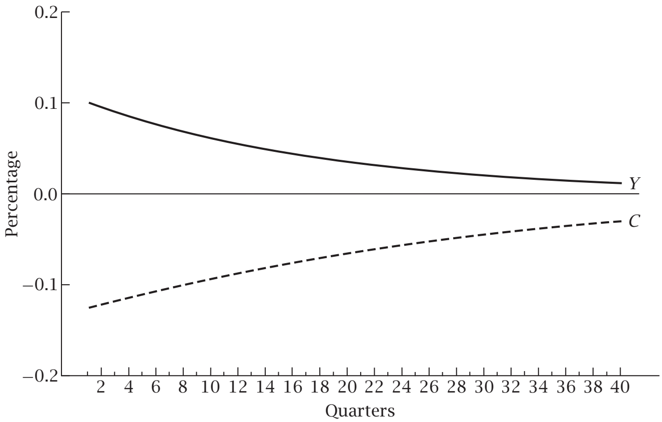


FIGURE 5.6 The effects of a 1 percent government-purchases shock on the paths of output and consumption

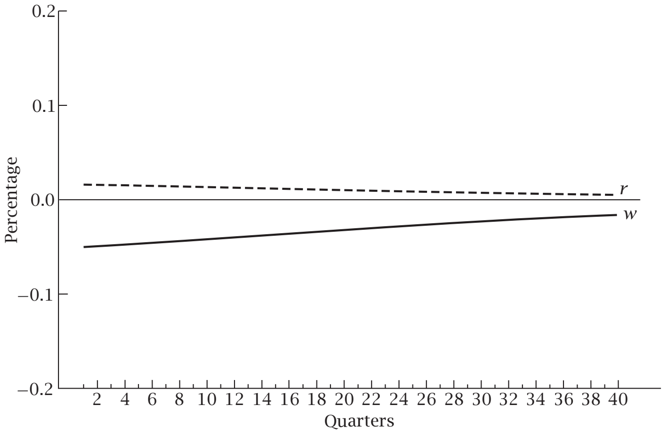


FIGURE 5.7 The effects of a 1 percent government-purchases shock on the paths of the wage and the interest rate

Romer RBC (ch.5) and Dynare

1. Use a numerical software Dynare to simulate Romer's real-business-cycle model of Chapter 5.
2. Construct our own versions of Figure 5.2 through Figure 5.7; compare the correlations and autocorrelations of the variables in stochastic simulations to those of the actual U.S. economy.
3. To facilitate solution in Dynare: simplify the model by expressing it in terms of log-deviations from the model's steady-state values, then calculate a linear approximation of the model's equations near the steady state in terms of these log-deviations.
4. With the log-linearization, you will use Dynare to perform various deterministic and stochastic simulations[1].

[1] Slides based on Jeff Parker (Reed College) course material, with permission.

Preparing the model for simulation with Dynare

Approximate deviations of variables around the steady state

- Ignore changes in the steady-state values themselves over time.

In Romer's model, the population N grows at a constant rate n and never deviates from its steady-state path.

- Deviations from steady-state never arise from changes in population
  - We can ignore population growth in the population.
- Without loss of generality, we set the constant value of the population at $N = 1$ (no per-capita variables)

Preparing the model for simulation with Dynare

- Convert the model to a log-linear approximation to deviations from the steady state

Notation:

* lower-case letters to represent the logs of capital-letter variables.

Some minor notation changes from Romer's Ch. 5.

- For example, $k \equiv \ln K$.
- $c \equiv \ln C$ and $l \equiv \ln L$;
    - Romer's Chapter 5 uses c and l to refer to C/N and L/N.

Preparing the model for simulation with Dynare

Other modifications to Romer's equations.

1. We use K_t to refer to the capital stock at the end of period t rather than (as Romer uses it) the beginning.

- This is just a change in timing convention that conforms better to Dynare's requirements; it does not change the structure of the model at all.

2. To reserve lower-case letters for logs, we denote the real wage by W and its log by w.

Preparing the model for simulation with Dynare

3. We define R_t to be the "gross" return on capital, which is one plus the net return that Romer calls r_t.

- Intuitively, the gross return on a bond includes the repayment of the bond plus the interest earned.
- Following our logarithmic-notation convention, we will define r_t to be $r_t \equiv \ln R_t \simeq R_t - 1$, so r_t is (within our approximation) the same as Romer's variable.

4. Because we will need g to be the log of G, we will denote the growth rate of productivity (Romer's g) as γ, with $e^{\gamma} \simeq 1 + \gamma$.

The Romer model in our notation

State the equations of the Romer model with these notation changes.

$$Y_t = K_{t-1}^{\alpha}(A_t L_t)^{1-\alpha} \quad (33)$$

$$K_t = (1-\delta)K_{t-1} + Y_t - C_t - G_t \quad (34)$$

Equation (33) is Romer's (5.1), the production function.

- The only change here is that the beginning-of-period capital stock is now K_{t-1} rather than K_t.

Equation (34) is Romer's (5.2), the capital-accumulation equation.

- The only change is the adjustment of the time subscripts of capital.

The Romer model in our notation

$$W_t = (1-\alpha)\left(\frac{K_{t-1}}{A_t L_t}\right)^{\alpha} A_t \tag{35}$$

$$R_t = \alpha\left(\frac{A_t L_t}{K_{t-1}}\right)^{1-\alpha} + (1-\delta) \tag{36}$$

Equation (35) is Romer's (5.3): wage equal to the marginal product of labor.

- capital W for the wage (reserving lower-case w for its log).

Equation (36) is Romer's (5.4): interest rate equal to the marginal product of capital minus the depreciation rate.

- We have added one to both sides of (5.4) to get (36):
  - R_t on the left is $e^{r_t} \cong 1 + r_t$, and the term $-\delta$ on the end of the right hand expression has changed to $1 - \delta$.

The Romer model in our notation

$$\frac{1}{C_t} = e^{-\rho} E_t \left(\frac{R_{t+1}}{C_{t+1}} \right) \quad (37)$$

$$bC_t = (1 - L_t) W_t \quad (38)$$

Equation (37) is Romer's (5.23): the Euler equation for intertemporal consumption.

- capital C because our variables are already in per-capita terms (and we want lower-case c to be its log) and our $R_t = 1 + r_t$.

Equation (38) is Romer's (5.26): equates the marginal utility of consumption to marginal utility of leisure.

- C and W are in capitals rather than Romer's lower-case, but otherwise the equations are identical.

The Romer model in our notation

$$\ln A_t \equiv a_t = \gamma t + \tilde{a}_t \quad \text{and} \quad \tilde{a}_t = \rho_A \tilde{a}_{t-1} + \varepsilon_{A,t} \tag{39}$$

$$\ln G_t \equiv g_t = \bar{g} + \gamma t + \tilde{g}_t \quad \text{and} \quad \tilde{g}_t = \rho_G \tilde{g}_{t-1} + \varepsilon_{G,t} \tag{40}$$

Equations (39) and (40): equations of motion for technology and government spending, corresponding to Romer's (5.8) – (5.11).

- The equations are identical except for introducing lower-case a and g as the logs of A and G, setting $n = 0$ (because we ignore population growth) and replacing Romer's growth rate g by γ.

Insights into how the model is solved

- Equations $(33) - (40)$ constitute eight equations in Y, K, L, A, W, R, C, and G, including: lags of K, expected future values of C, R.

Suppose that we consider an economy that starts at time zero in a steady state with a given, known value of K_0.

- Whatever shocks hit the system at time 0, we expect that the economy will re-converge to a steady state after some large number of periods, say T.
- We can iterate the evolution of capital forward through the T periods from its initial value K_0 using equation (34) (and the other equations of the model).

Insights into how the model is solved

Because we know that the shock will have died out by period $T+1$, we assume that C and R will be back to their steady-state values by then.

- This allows us to solve the model as $8 \times T$ equations in the $8 \times T$ variables corresponding to periods 1 through T.

This solution is much more feasible if the equations of the model are linear (not to mention making handing the expectations feasible).

Next steps:

- Calibration and Parametrization
- Express the model as a linear approximation of deviations from the steady state.

Calibration of the model

Values for the parameters of a model (Romer's Section 5.7)

Assuming that each period is one quarter:

- $\alpha = 1/3$, which corresponds roughly to capital's share of income,
- $\gamma = 0.005$, which is a 2% per year growth rate of productivity,
- $\delta = 0.025$, a 10% per year rate of depreciation,
- $\rho_A = \rho_G = 0.95$: both productivity and government spending return to their steady-state paths at a rate of 5% per quarter.

Calibration of the model

The other parameters (ρ, b, and $\bar{g}$) will be set in such a way that:

- the steady-state government/output ratio is $(G/Y)^* = 0.2$,
- the steady-state interest rate is $r^* = 0.015$ (per quarter), and
- the steady-state $L^* = 1/3$
  (our representative agent works one-third of the time).

Some key steady-state values

The long-run behavior of this model is essentially identical to that of the Ramsey growth model

Y, K, C, G, W, and A should all grow at constant rate γ on their steady-state growth paths

R and L should be constant (trendless) in the steady state.

- The steady-state values are $R^* = 1 + r^* = 1.015$ and $L^* = 1/3$.

Some key steady-state values

Because many of the variables are growing in the steady state, it makes sense to think about **ratios among variables that might be constant.**

- *Much like the K/AL variable we used in the Solow and Ramsey growth models.*

It will be convenient later to know the steady-state values of three ratios: A_t/K_{t-1}, Y_t/K_{t-1}, and C_t/Y_t, denoted respectively as

$$\left(\frac{A}{K_{-1}}\right)^*, \left(\frac{Y}{K_{-1}}\right)^* \text{ and } \left(\frac{C}{Y}\right)^*$$

Some key steady-state values

The steady-state value of the interest rate in terms of the model's parameters from (37).

- In the steady state, consumption is growing at rate γ so $C_{t+1} = e^{\gamma} C_t$. This means that

$$\begin{aligned} \frac{1}{C_t} &= e^{-\rho} \frac{R^*}{e^{\gamma} C_t} \\ &\Downarrow \\ R^* &= e^{\rho+\gamma} \end{aligned} \tag{41}$$

Given our calibrations of steady-state r and γ, this allows us to determine ρ.

$$R^* \cong 1.015 = e^{\rho+0.005}, \quad or \quad \rho = 0.0099 \cong 0.01$$

Some key steady-state values

From equation (36), in the steady state,

$$R^* = e^{\rho+\gamma} = \alpha \left(\frac{A}{K_{-1}} L^* \right)^{1-\alpha} + (1-\delta)$$

Rearranging

$$\left(\frac{A}{K_{-1}} \right)^* = \frac{1}{L^*} \left[\frac{e^{\rho+\gamma} - (1-\delta)}{\alpha} \right]^{\frac{1}{1-\alpha}} \cong \frac{1}{L^*} \left[\frac{\rho+\gamma+\delta}{\alpha} \right]^{\frac{1}{1-\alpha}} \qquad (42)$$

Recall that we are setting $L^* = 1/3$ as a calibration parameter

- Equation (42): a steady-state value for the ratio of A to last period's K that depends only on parameters we assume we know.

Some key steady-state values

Problem

Show that the steady-state value of Y_t/K_{t-1} *and* C_t/Y_t *are, respectively*

$$\left(\frac{Y_t}{K_{t-1}}\right) = \frac{\rho+\gamma+\delta}{\alpha}$$

$$\left(\frac{C_t}{Y_t}\right) = 1-\left(\frac{G}{Y}\right)^{*}-\alpha\left[\frac{\delta+\gamma}{\rho+\gamma+\delta}\right]$$

Log-linearization of deviations from the steady state

To log-linearize the model, we introduce additional notation.

- For a variable X_t, the log-deviation is defined as

$$\tilde{x}_t \equiv \ln X_t - \ln X_t^* = x_t - x_t^*$$

  where X_t^* the value that X would have if the economy were in a steady state at t.

The goal of log-linearization:

- **Express the equations of the model as a linear function of these log-deviation variables.**
- Some equations are already linear in the logs of the variables.
- Others are not linear in logs and must the approximated using Taylor series.

Log-linearization of deviations from the steady state

We will work through equations (33), (34), and (37) here.

Equations (39) and (40) are already in terms of log-deviations.

Problem

Develop log-linear versions of equations (35), (36), *and* (38).

Log-linearization of deviations from the steady state

Equation (33) is an example of an equation that is already linear in logs.

- Such equations do not require approximation and are easy to handle.

Taking the logs of both sides of (33) yields (and its steady state version)

$$\begin{aligned} y_t &= \alpha k_{t-1} + (1-\alpha)(a_t + l_t) && (43) \\ y_t^* &= \alpha k_{t-1}^* + (1-\alpha)(a_t^* + l_t^*) && (44) \end{aligned}$$

Subtracting (44) from (43) gives us

$$\tilde{y}_t = \alpha \tilde{k}_{t-1} + (1-\alpha)(\tilde{a}_t + \tilde{l}_t) \qquad (45)$$

Equation (45) is linear in the log-deviation (tilde) variables, so it is the first equation of our log-linearized system.

Log-linearization of deviations from the steady state

Equation (34) is linear in the levels of the variables, not in their *logs*.

- The Taylor approximation of equation (34) will be accomplished by transforming the equation into two additive parts, then taking the approximation of each part.

We begin by dividing both sides of (34) by K_{t-1} to get

$$\begin{aligned}\frac{K_t}{K_{t-1}} &= (1-\delta) + \frac{Y_t}{K_{t-1}} - \frac{C_t}{K_{t-1}} - \frac{G_t}{K_{t-1}} \\ \frac{K_t}{K_{t-1}} - (1-\delta) &= \frac{Y_t}{K_{t-1}}\left(1 - \frac{C_t}{Y_t} - \frac{G_t}{Y_t}\right)\end{aligned}$$

Log-linearization of deviations from the steady state

$$\frac{K_t}{K_{t-1}} - (1-\delta) = \frac{Y_t}{K_{t-1}}\left(1 - \frac{C_t}{Y_t} - \frac{G_t}{Y_t}\right)$$

This expression is convenient because we know the steady-state values of $\frac{K_t}{K_{t-1}}$, $\frac{Y_t}{K_{t-1}}$, $\frac{C_t}{Y_t}$, and $\frac{G_t}{Y_t}$

- K grows at rate γ in the steady state, so the steady-state value of $\frac{K_t}{K_{t-1}}$ is $e^{\gamma} \cong 1+\gamma$,
- The steady-state value of $\left(\frac{Y_t}{K_{t-1}}\right)$ is $\frac{\rho+\gamma+\delta}{\alpha}$,
- The steady-state value of $\left(\frac{C_t}{Y_t}\right)$ is $1 - \left(\frac{G}{Y}\right)^* - \alpha\left[\frac{\rho+\gamma}{\rho+\gamma+\delta}\right]$.

Log-linearization of deviations from the steady state

Taking logs of both sides of this equation,

$$\begin{aligned} \ln\left[\exp(\Delta k_t) - (1-\delta)\right] &= y_t - k_{t-1} \\ &\quad + \ln\left[1 - \exp(c_t - y_t) - \exp(g_t - y_t)\right] \end{aligned} \tag{46}$$

where

$$\begin{aligned} \ln\left(\frac{K_t}{K_{t-1}}\right) &= \ln K_t - \ln K_{t-1} = k_t - k_{t-1} \equiv \Delta k_t \\ \text{so} \quad (K_t / K_{t-1}) &= \exp(\Delta k_t) \end{aligned}$$

and

$$\begin{aligned} \ln\left(\frac{C_t}{Y_t}\right) = \ln(C_t) - \ln(Y_t) &= c_t - y_t \\ \text{so} \quad (C_t / Y_t) &= \exp(c_t - y_t) \end{aligned}$$

Log-linearization of deviations from the steady state

$$\begin{aligned}\ln\left[\exp(\Delta k_t) - (1-\delta)\right] &= y_t - k_{t-1} \\ &\quad + \ln\left[1 - \exp(c_t - y_t) - \exp(g_t - y_t)\right]\end{aligned}$$

Equation (46) expresses equation (34) in terms of the logs of the variables, but it is highly nonlinear.

- We approximate nonlinear parts of (46) with first-order Taylor series

Log-linearization of deviations from the steady state

$$\begin{aligned}\ln\left[\exp(\Delta k_t) - (1-\delta)\right] &= y_t - k_{t-1} \\ &\quad + \ln\left[1 - \exp(c_t - y_t) - \exp(g_t - y_t)\right]\end{aligned}$$

LHS: Define the function

$$f_1(\Delta k_t) = \ln\left[\exp(\Delta k_t) - (1-\delta)\right]$$

Approximate $f_1(\Delta k_t)$ in the neighborhood around the steady state by

$$f_1(\Delta k_t) = f_1(\gamma) + f_1'(\gamma)(\Delta k_t - \gamma)$$

where the growth rate γ is known to be the steady state value of Δk_t.

Log-linearization of deviations from the steady state

Using the chain rule,

$$f_1'(\Delta k_t) = \frac{1}{\exp(\Delta k_t) - (1-\delta)} \exp(\Delta k_t)$$

Evaluating this expression at the steady-state value $\Delta k_t = \gamma$ and taking advantage of the approximation $\exp(\gamma) \cong 1 + \gamma$ gives

$$f_1'(\gamma) \cong \frac{1+\gamma}{(1+\gamma) - (1-\delta)} = \frac{1+\gamma}{\delta+\gamma} \tag{47}$$

Log-linearization of deviations from the steady state

RHS: Define the function

$$f_2\left((c_t - y_t), (g_t - y_t)\right) = \ln\left[1 - \exp(c_t - y_t) - \exp(g_t - y_t)\right]$$

For this function of two variables, the first-order Taylor series approximation is

$$\begin{aligned} f_2\left((c_t - y_t), (g_t - y_t)\right) \cong\ & f_2\left((c - y)^*, (g - y)^*\right) \\ & + \frac{\partial f_2(\cdot)}{\partial(c_t - y_t)}\left[(c_t - y_t) - (c - y)^*\right] \\ & + \frac{\partial f_2(\cdot)}{\partial(g_t - y_t)}\left[(g_t - y_t) - (g - y)^*\right] \end{aligned}$$

where $(c - y)^*$ and $(g - y)^*$ are the steady-state values.

- $(g - y)^*$: log of the calibration parameter $(G/Y)^*$

Log-linearization of deviations from the steady state

From the expression of $\left(\frac{C_t}{Y_t}\right)$ in steady state

$$(c-y)^{*}=\ln\left(\frac{C}{Y}\right)^{*}=\ln\left\{1-\left(\frac{G}{Y}\right)^{*}-\alpha\left[\frac{\delta+\gamma}{\rho+\gamma+\delta}\right]\right\}$$

As before, our interest in deviations around the steady state focuses our attention on the last two terms of the Taylor approximation.

Computing the partial derivative with respect to (c_t-y_t),

$$\begin{aligned}\frac{\partial f_2(\cdot)}{\partial(c_t-y_t)} &= \frac{1}{1-\exp(c_t-y_t)-\exp(g_t-y_t)}\left(-\exp(c_t-y_t)\right)\\ &= -\frac{(C_t/Y_t)}{1-(C_t/Y_t)-(G_t/Y_t)}\end{aligned}$$

Log-linearization of deviations from the steady state

At the steady-state value gives

$$\begin{aligned}
\frac{\partial f_2}{\partial(c_t - y_t)}\left((c-y)^*, (g-y)^*\right) &= -\frac{(C/Y)^*}{1-(C/Y)^*-(G/Y)^*} \\
&= 1-\frac{1-(G/Y)^*}{\alpha\left[\frac{\delta+\gamma}{\rho+\gamma+\delta}\right]} \\
&\equiv 1-\frac{\rho+\gamma+\delta}{\alpha(\delta+\gamma)}\left[1-(G/Y)^*\right]
\end{aligned}$$

Since $(G/Y)^*$ is a calibrated parameter, this can be evaluated numerically at the steady state in terms of the parameters of the model.

Following similar logic, we can show that

$$\frac{\partial f_2}{\partial(g_t - y_t)}\left((c-y)^*, (g-y)^*\right) = -\frac{\rho+\gamma+\delta}{\alpha(\delta+\gamma)}(G/Y)^*$$

Log-linearization of deviations from the steady state

Express equation (46) in terms of deviations around the steady state. Subtracting the steady-state values yields

$$\left\{ \begin{array}{c} \ln\left[\exp(\Delta k_t) - (1-\delta)\right] \\ -\ln\left[\exp(\gamma) - (1-\delta)\right] \end{array} \right\} = \begin{array}{l} (y_t - y^*) - (k_{t-1} - k^*_{t-1}) \\ + \ln\left[\begin{array}{c} 1 - \exp(c_t - y_t) \\ - \exp(g_t - y_t) \end{array} \right] \\ - \ln\left[1 - (C/Y)^* - (G/Y)^*\right] \end{array}$$

or

$$\begin{aligned} f_1(\Delta k_t) - f_1(\gamma) &= (y_t - y^*) - (k_{t-1} - k^*_{t-1}) \\ &\quad + f_2\left[(c_t - y_t), (g_t - y_t)\right] \\ &\quad - f_2\left[(c-y)^*, (g-y)^*\right] \end{aligned}$$

Log-linearization of deviations from the steady state

Substituting from the Taylor-series approximations for the expression on the left and the expression on the second line of the right:

$$
\begin{aligned}
\frac{1+\delta}{\delta+\gamma}(\Delta k_t - \gamma) \quad = \quad & (y_t - y^*) - (k_{t-1} - k^*_{t-1}) \\
& + \left\{ \begin{array}{c} \left(1 - \frac{\rho+\gamma+\delta}{\alpha(\delta+\gamma)}\left[1 - (G/Y)^*\right]\right) \\ \left[(c_t - c^*_t) - (y_t - y^*_t)\right] \end{array} \right\} \\
& + \left\{ \begin{array}{c} \left(-\frac{\rho+\gamma+\delta}{\alpha(\delta+\gamma)}(G/Y)^*\right) \\ \left[(g_t - g^*_t) - (y_t - y^*_t)\right] \end{array} \right\}
\end{aligned}
$$

Log-linearization of deviations from the steady state

Or, in terms of the derivations

$$
\begin{aligned}
\frac{1+\delta}{\delta+\gamma}(\tilde{k}_t-\tilde{k}_{t-1}) &= \left(\tilde{y}_t-\tilde{k}_{t-1}\right) \\
&+\left(1-\frac{\rho+\gamma+\delta}{\alpha(\delta+\gamma)}\left[1-(G/Y)^*\right]\right)(\tilde{c}_t-\tilde{y}_t) \\
&-\left(\frac{\rho+\gamma+\delta}{\alpha(\delta+\gamma)}(G/Y)^*\right)\left[(\tilde{g}_t-\tilde{y}_t\right]
\end{aligned}
$$

Log-linearization of deviations from the steady state

Collecting terms in $\tilde{y}_t$ and $\tilde{k}_{t-1}$

$$\begin{aligned}(1+\gamma)\tilde{k}_t &= (1-\delta)\tilde{k}_{t-1} + \left(\frac{\rho+\gamma+\delta}{\alpha}\right)\tilde{y}_t \\ &\quad + \left[(\gamma+\delta) - \left(\frac{\rho+\gamma+\delta}{\alpha}\right)\left[1-\left(\frac{G}{Y}\right)^*\right]\right]\tilde{c}_t \\ &\quad - \left(\frac{\rho+\gamma+\delta}{\alpha}\right)\left(\frac{G}{Y}\right)^*\tilde{g}_t \qquad (48)\end{aligned}$$

Equation (48) is an approximation of equation (34) that is linear in terms of the log-deviations from the steady state.

- Note again that all of the coefficients appearing in (48) and $(G/Y)^*$ are parameters that we assume to be known.

Log-linearization of deviations from the steady state

Log-linearization of **equation** (37), which needs special attention because of the expectation operator on the right-hand side.

If we take logs of both sides, we get

$$-c_t = -\rho + \ln\left[E_t\left(\frac{R_{t+1}}{C_{t+1}}\right)\right] \tag{49}$$

- If we could just take the expectation operator outside the log operator, this could easily be reduced to an expression in $\tilde{c}_t$, $\tilde{r}_{t+1}$ and $\tilde{c}_{t+1}$.
- However, the log of an expectation is not equal to the expectation of the log, so it is not so simple.

In order to simplify (49) we must make a simple assumption about the distribution of the random shocks.

Log-linearization of deviations from the steady state

- If the ε shocks both follow the normal probability distribution, then all of the log variables in the linearized system will also be normal random variables and the capital-letter variables will follow the log-normal distribution.

If R and C are distributed log-normally, then

$$\ln\left[E_t\left(\frac{R_{t+1}}{C_{t+1}}\right)\right] = E_t(r_{t+1} - c_{t+1}) + \zeta$$

where ζ (zeta) is a constant.

- Because ζ is the same in the steady state as outside, it will cancel when we take deviations from the steady state, so we need not evaluate it.

Log-linearization of deviations from the steady state

Applying this solution to the expectation problem and taking deviations from the steady state in (49) yields

$$-\tilde{c}_t = E_t \tilde{r}_{t+1} - E_t \tilde{c}_{t+1} \tag{50}$$

Equation (50) **is our log-deviation version of equation** (37).

Log-linearization of deviations from the steady state

Problem

Using the methods in the derivation of equations (45), (48), derive an expression
(a) for equation (35) in terms of $\tilde{w}_t$, $\tilde{k}_{t-1}$, $\tilde{a}_t$, *and* $\tilde{l}_t$;
(b) for equation (36) in $\tilde{r}_t$, $\tilde{k}_{t-1}$, $\tilde{a}_t$, *and* $\tilde{l}_t$; *and*
(c) for equation (38) in $\tilde{w}_t$, $\tilde{c}_t$, *and* $\tilde{l}_t$.

Preamble section

The definitions of the variables are straightforward.

I've used lower-case letters corresponding to the notation in the text (but without tildes).

Similarly, the parameters are named in ways that correspond to the equations of the model.

The parameter values correspond to those at the beginning of Romer's Section 5.7.

Preamble section

```
// Define the endogenous variables
var k a l c y w r g;

// Define the exogenous variables (shocks)
varexo eps_g eps_a;

// Define the parameters of the model
parameters alpha rhoa rhog rho lstar gamma delta gystar;
```

Preamble section

```
// Set the values of the parameters to match Romer's calibration
alpha = 1/3;
rhoa = 0.95;
rhog = 0.95;
rho = 0.01;
lstar = 1/3;
gamma = 0.005;
delta = 0.025;
gystar = 0.2;
```

Model section

The equations are entered in standard computer-algebraic notation, with +, –, *, /, and ^ as the principal operators and nested sets of parentheses used to indicate the order of operation.

- Note that it is not necessary to "solve" the individual equations to isolate an endogenous variable alone on the left-hand side.

Suffix parentheses are used to indicate leads and lags, with x(–1) denoting last period's value of x and x(+1) indicating today's expectation of next year's value of x.

Equations can span more than one line of the program, but be sure that each equation ends with a semi-colon.

Model section

```
// ***Model section***
model(linear);

// Equation (50) from slides. Expected future value of x is x(+1)
-c = r(+1) - c(+1);

// Equation (48) from slides. Lags of x denoted by x(-1)
(1+gamma)*k = ((rho+gamma+delta)/alpha)*y +
(1-delta)*k(-1) +
((gamma+delta)-((rho+gamma+delta)*(1-gystar)/alpha))*c -
(gystar*(rho+gamma+delta)/alpha)*g;

// Equation (45) from slides.
y = alpha*k(-1) + (1-alpha)*(a + l);
```

Model section

You will need to add Dynare code for the three linearized log-deviation equations that you derive in Homework 1 based on equations (35), (36), and (38).

// YOU WILL NEED TO ADD THREE EQUATIONS HERE
// Equations (35), (36) and (38)

// Equations (39) and (40) from above
```
a = rhoa*a(-1) + eps_a;
g = rhog*g(-1) + eps_g;
```
end;

Steady-state section

The variables of our Dynare model are all log-deviations from the steady state, thus their steady-state values are simply zero

- When the economy is in the steady state there are no deviations.

This makes the steady-state section trivial and it need not be changed for your simulations.

The **steady** and **check** commands simply direct Dynare to verify that these are indeed a steady state and that certain dynamic-stability conditions hold.

Steady-state section

```
// *** Steady-state section ***
initval;  c = 0;
l = 0;
k = 0;
y = 0;
g = 0;
a = 0;
w = 0;
r = 0;
end;
steady;
check;
```

Shocks section

This section defines the shocks that you want to simulate.
Two kinds of simulations: deterministic and stochastic.

- In a deterministic simulation, you set all shocks to zero except those specified in the shocks section and you use the **simul** command in the computation section.
  - The code below is set up to run a deterministic simulation in which the value of epsa (ε_A) is set to $+0.01$ (one percent) in period one.
- All other shocks (the government spending shock) and values of epsa in periods other than period one are set to zero.

This command should generate simulations that match those shown in Romer's Figures 5.2 through 5.4. To perform simulations of the government spending shock, you could change epsa to epsg in the var command of the shocks section.

Shocks section

```
// *** Shocks section ***
// *** Deterministic simulation ***
shocks;
var eps_a;
periods 1;
values 0.01;
%var eps_g;
%periods 1;
%values 0.01;
end;
```

Shocks section

Stochastic simulations involve drawing a random value for each shock in each period from a normal probability distribution with specified standard deviations and correlations for the shocks.

- For example, to simulate the model with random shocks to both epsa and epsg each having a standard deviation of 0.01 (one percent) and no correlation between them, you would enter the shocks section as follows:

```
// *** Shocks section for stochastic simulation***  shocks;
var epsa; stderr 0.01;
var epsg; stderr 0.01;
end;
```

Shocks section

The fact that you have not entered a correlation value means that Dynare sets it to zero.

In fact, any standard deviation or correlation that is not explicitly set in the shocks section is assumed to be zero.

So if you left out the var epsg line, you would get a simulation with the technology shock varying randomly but the government-spending shock set to zero through-out (because a random variable with a standard deviation of zero is just a constant).

Computation section

The computation section directs Dynare to simulate the model and create specific output elements that you want.

- As with the shocks section, the computation section will depend on whether you are doing a deterministic or stochastic simulation.

```
// *** Computation section ***
simul(periods=200);
```

The periods=200 option tells Dynare to assume that the model will be back in (or sufficiently close to) the steady-state after 200 periods (quarters).

Computation section

Setting this option to a value that is too small will cause the simulation to be incorrect.

Examine your results closely to see if there is a "jump" at the end of the simulations. If there is, then you haven't set this option high enough.

There are many other options for the simul command that are described in the Dynare documentation.

Computation section

If you are doing a stochastic simulation, you will use the **stoch_simul** command.

- Like the **simul** command, it takes a periods option that determines the number of periods in the simulation.

In a stochastic simulation, you may want to set the number of periods to be much larger (2000?) so that the summary statistics of the simulation can be averaged over a larger number of periods.

Producing graphic output

Dynare and Octave/MATLAB are capable of graphing your impulse-response functions (from either a deterministic or stochastic simulation) and your simulated time series from a stochastic simulation.

```
// *** Some MATLAB plot commands
subplot(3,3,1);
plot(100*oo_.endo_simul(1,2:40));
title('k');
subplot(3,3,2);
plot(100*oo_.endo_simul(2,2:40));
title('a');

.....
```