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Writing Sample

# A survey on Dynamic Equilibrium with Limited Commitment

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# Chapter 1

# Introduction

This material consists on a survey about some papers pertinent to my current research. The objective of my research, on the other hand, is to develop an alternative algorithm to find the stationary equilibria of the Azariadis and Kaas (2012) model. This model is a generalization of the Alvarez and Jermann (2000) framework, where individuals who default have a chance $\mu \in [0, 1]$ of regaining access to the asset market in the next periods. Therefore, the expected duration of exclusion from the asset market in this environment is $1/\mu \leq +\infty$.

Since this model has an endogenous participation constraint, like Hellwig and Lorenzoni (2009), we can not implement Alvarez and Jermann's (2001) algorithm to find its equilibria. Azariadis and Kaas (2012) use, instead, an approximation to compute all the stationary equilibria of their model. Our objective, then, is to use an alternative method to find these equilibria, which consists in implementing an algorithm similar to the ones developed by Hugget (1993) and Aiyagari (1994).

# Chapter 2

# Credit limits with "high" costs of default (permanent exclusion from the asset market)

## 2.1 Introduction

In this chapter, we present the definition of Kehoe and Levine (1993) equilibrium for an economy with only one nondurable good, and present a survey on the Alvarez and Jermann (2000) paper.

We present a theorem, proved by Alvarez and Jermann (henceforth A-J) , which guarantees that the allocations and Arrow prices derived from an A-J equilibrium are identical to the allocations and prices of a Kehoe and Levine (henceforth K-L) equilibrium, provided that some specific conditions are satisfied. Since every K-L equilibrium is constrained efficient (i.e., it is efficient in the class of all allocations that satisfy the participation and feasibility constraints), this theorem gives us as a corollary conditions under which the A-J equilibrium is efficient, which is a version of the first welfare theorem for the A-J equilibrium.

Finally, we state some consequences that imperfect commitment, as modeled by A-J (2000), may have on asset prices. More precisely, we present the intuition given by A-J (2000) to why the interest rate in this economy is lower than the interest rate of a corresponding Arrow economy without solvency constraints.

## 2.2 Environment

In this section, we specify the environment used to introduce the concept of K-L equilibrium and A-J equilibrium. This environment will also be used to introduce variants of the A-J equilibrium in the next chapter, such as the Sequential equilibrium with permanent exclusion from borrowing (Hellwig and Lorenzoni (2009)). Throughout this section, we use the same notation as A-J (2000).

Suppose there are $\mathbf{I} = \{1, 2, \cdots, I\}$ infinitely lived agents in the economy, indexed by $i$. At each period $t \in \{0, 1, 2, \cdots\}$, there is an aggregate shock $z_t \in Z$, where $Z$ is finite. We assume the process of shocks $\{z_t\}$ follows a Markov process with transition probability $\Pi$. Following conventional notation, we define $z^t$ as a history of shocks $(z_0, z_1, \cdots, z_t)$ with length $t$, and $Z^t$ as the set of all possible histories with length $t$. The partial order $\succeq$ defined on the set $\{z^t \in Z^t; t = 0, 1, 2, \cdots\}$ is such that $z^{t'} \succeq z^t$ iff $t' \geq t$ and $z^{t'}$ is a possible continuation of history $z^t$.

Assume that there is only one nondurable good in the economy. At each period $t$, individual $i \in \mathbf{I}$ is endowed with $e_i(z_t) > 0$ units of the nondurable good, whenever the current aggregate shock is $z_t$. The stochastic process of consumption is defined as $\{c_i\} \equiv \{c_i(z^t); t \geq 0 \wedge z^t \in Z^t\}$. Individuals are assumed to have conventional time separable expected utility function, and their continuation utility from consuming $\{c_i\}$ at date $t$ history $z^t$ is given by

$$U(c)(z^t) \equiv \sum_{s=t}^{\infty} \sum_{z^s \in Z^s} \beta^t u(c_s(z^s))\pi(z^s|z^t),$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, strictly concave and $C^1$, and where $\beta \in (0, 1)$ is the time invariant discount factor[1].

## 2.3 K-L equilibrium

The K-L equilibrium of this economy is defined as follows:

**Definition 2.3.1** *A K-L equilibrium consists of quantities* $\{c_i\}$ *and prices* $\{Q_0(z^t|z_0)\}$, *such that*

*i)* $\{c_i\}$ *is resource feasible:*

$$\sum_i c_{i,t}(z^t) = \sum_i e_{i,t}(z^t) \quad \forall t,\, \forall z^t$$

*and*

*ii) given* $\{Q_0(z^t|z_0)\}$, *for each* $i$ $\{c_i\}$ *maximizes*

$$U(c_i)(z_0), \tag{2.1}$$

*subject to*

$$\sum_{t\geq 0} \sum_{z^t \in Z^t} c_{i,t}(z^t)Q_0(z^t|z_0) \leq \sum_{t\geq 0} \sum_{z^t \in Z^t} e_{i,t}(z^t)Q_0(z^t|z_0) \tag{2.2}$$

$$U(c_i)(z^t) \geq U(e_i)(z^t) \quad \forall t,\, \forall z^t. \tag{2.3}$$

In other words, a K-L equilibrium are A-D prices and feasible allocations such that, given the A-D prices, the equilibrium allocation maximizes each individual

[1] In order to allow growth in aggregate consumption, A-J assume that individuals have a stochastic discount factor instead of assuming that the discount factor is constant and equal to $\beta$. In the Appendix, we show that A-J specification with stochastic discount factor is indeed equivalent to an economy with constant discount factor that experiences growth in aggregate endowment.

continuation utility starting from state $z_0$ subject to his budget constraint (eq. 2.2) and to his participation constraint (eq. 2.3).

Now notice that every K-L equilibrium is a standard A-D equilibrium with restrictions on the consumption possibility set. Therefore, the *First Welfare Theorem* for an A-D economy also applies to a K-L equilibrium, so we have that every K-L equilibrium allocation is constrained efficient (i.e., it is efficient in the class of all allocations that satisfy the participation and resource-feasibility constraints).

The corresponding Arrow prices (pricing kernel) of the K-L equilibrium are defined as

$$q_{0,t}(z^t, z') = \frac{Q_0(z^t, z'|z_0)}{Q_0(z^t|z_0)}.$$

We can prove that these prices are determined by the marginal valuations of unconstrained agents. This is formally stated in the following proposition:

**Proposition 2.3.1** *(Alvarez and Jermann (2000), page 784) Let $\{c_i\}$, $i = 1, \cdots, I$, and $\{Q_0\}$ be the allocations and A-D prices corresponding to a K-L equilibrium. Let $q_0$ be the corresponding Arrow prices. Then,*

$$q_{0,t}(z^t, z') = \max_{i \in \mathbf{I}} \left\{ \beta \frac{u'(c_{i,t+1}(z^t, z'))}{u'(c_{i,t}(z^t))} \pi(z'|z_t) \right\}, \tag{2.4}$$

*and if*

$$U(c_i)(z^t, z') > U(e_i)(z^t, z'),$$

*then*

$$q_{0,t}(z^t, z') = \beta \frac{u'(c_{i,t+1}(z^t, z'))}{u'(c_{i,t}(z^t))} \pi(z'|z_t).$$

**Proof:** See the appendix. ■

## 2.4 A-J equilibrium

**Definition 2.4.1** *(Alvarez and Jermann (2000), page 780) An Arrow equilibrium with solvency constraints $\{B_i\}$ are initial conditions $\{a_{i,0}\}$, quantities $\{c_i, a_i\}$ and Arrow prices $\{q\}$ such that:*

*i) given $\{q\}$, for each $i$, $\{c_i, a_i\}$ solves*[2]

$$J_{i,t}(a, z^t) = \max_{c, \{a_{z'}\}_{z' \in Z}} \left\{ u(c) + \beta \sum_{z' \in Z} J_{i,t+1}(a_{z'}, (z^t, z'))\pi(z', z_t) \right\}, \tag{2.5}$$

[2]Notice that the value function is indexed by $t$ because the dimension of the state variable $z^t$ increases with $t$.

*subject to*

$$e_{i,t}(z^t) + a = \sum_{z' \in Z} a_{z'} q_t(z^t, z') + c, \tag{2.6}$$

$$a_{z'} \geq B_{i,t+1}(z^t, z') \quad \forall z' \in Z \tag{2.7}$$

*and*

*ii) markets clear:*

$$\sum_i c_{i,t}(z^t) = \sum_i e_{i,t}(z^t) \quad \forall t, \, \forall z^t$$

$$\sum_{i \in \mathbf{I}} a_{i,t+1}(z^t, z') = 0 \quad \forall t, \, \forall z^t, \, \forall z'$$

For future reference, it is important to establish sufficient conditions for $\{c_i, a_i\}$ to solve consumer's i problem in the above equilibrium definition. These conditions are stated in the following proposition:

**Proposition 2.4.1** *(Alvarez and Jermann (2000), page 780) The following Euler and transversality conditions are sufficient for a maximum for problem 2.5:*

$$-u'(c_{i,t}(z^t))q_t(z^t, z') + \beta\pi(z', z)u'(c_{i,t+1}(z^t, z')) \leq 0 \tag{2.8}$$

*with equality if* $a_{i,t+1} > B_{i,t+1}(z^t, z')$ *and*

$$\lim_{t \to \infty} \sum_{z^t \in Z^t} \beta^t u'(c_{i,t}(z^t))[a_{i,t}(z^t) - B_{i,t}(z^t)]\pi(z^t | z_0) = 0. \tag{2.9}$$

**Proof:** See the Appendix. ■

Notice that the definition of the equilibrium above doesn't say anything about the possibility of default. The next step, then, consists in allowing agents to default, where the penalty from default consists in the permanent exclusion from trading in the asset market[3].

It can be show that if the solvency constraints in this economy are tight enough, then no one will ever choose to default in equilibrium. A-J works with *solvency constraints that are not too tight*, that is, with solvency constraints that are just tight enough to prevent default but allow as much risk sharing as possible. The formal definition of *not too tight solvency constraints* is given below:

**Definition 2.4.2** *(Alvarez and Jermann (2000), page 780) The solvency constraints* $\{B_i\}$ *of the Arrow equilibrium defined above are not too tight if*

$$J_{i,t+1}(B_{i,t+1}(z^{t+1}), z^{t+1}) = U(e_i)(z^{t+1}), \quad \forall t \geq 0, \, z^{t+1} \in Z^{t+1}. \tag{2.10}$$

[3]This kind of punishment is called in the literature as bilateral exclusion from the equity market, since the agent is not allowed to either by or sell assets, once he has been excluded from the asset market.

For future reference, we define an Arrow equilibrium with solvency constraints that are not too tight as an A-J equilibrium.

The next proposition shows that if solvency constraints are not too tight, then they only bind whenever the corresponding participation constraint of the K-L equilibrium binds.

**Proposition 2.4.2** *If $\{c_i, a_i\}$ is an A-J equilibrium allocation with solvency constraints $\{B_i\}$ that are not too tight, then for all $t$ and $z^t \in Z^t$,*

$$\begin{aligned} &U(c_i)(z^t) \geq U(e_i)(z^t) \quad \text{and} \\ &U(c_i)(z^t) = U(e_i)(z^t) \iff a_{i,t}(z^t) = B_{i,t}(z^t). \end{aligned}$$

**Proof:** See the appendix. ■

Finally, there is the concept of *high implied interest rates*. This condition ensures that the present value of aggregate endowment implied by a given allocation is finite, by requiring Arrow prices to be sufficiently low, (or equivalently, by requiring the implied interest rates to be sufficiently high).

Given the allocations $(\{c_i^*\})_{i\in\mathbf{I}}$, define

$$q_t^*(z^t, z') \equiv \max_{i\in\mathbf{I}} \left\{ \beta \frac{u'(c_{i,t+1}^*(z^t, z'))}{u'(c_{i,t}^*(z^t))} \pi(z'|z_t) \right\} \tag{2.11}$$

and

$$Q_0^*(z^t|z_0) = q_0^*(z_0, z_1) q_1^*(z_0, z_1, z_2) \cdots q_{t-1}^*(z^{t-1}, z_t). \tag{2.12}$$

**Definition 2.4.3** *(Alvarez and Jermann (2000), page 781) The implied interest rates for the allocation $\{c_i^*\}$ are high if*

$$\sum_{t\geq 0} \sum_{z^t\in Z^t} Q_0(z^t|z_0) \underbrace{\left( \sum_{i\in\mathbf{I}} c_{i,t}^*(z^t) \right)}_{e_t(z^t)} < +\infty. \tag{2.13}$$

Alvarez and Jerman (2000) define a version of the second welfare theorem for their definition of equilibrium, which we state (without proof) in the following proposition:

**Proposition 2.4.3** *(Alvarez and Jermann (2000), page 781) Any constrained efficient allocation that has high implied interest rates can be decentralized as an A-J competitive equilibrium with solvency constraints that are not too tight.*

According to the next proposition, there are always prices for which autarky is an A-J equilibrium with solvency constraints that are not too tight.

**Proposition 2.4.4** *(Alvarez and Jermann (2000), page 782) The quantity and prices*

$$c_{a,i,t}(z^t) = e_{i,t}(z^t), \quad a_{a,i,t+1}(z^t, z') \equiv B_{a,i,t}(z^t) = 0,$$
$$q_{a,t}(z^t, z') \equiv \max_{i \in \mathbf{I}} \left\{ \beta \frac{u'(e_{i,t+1}(z^t, z'))}{u'(e_{i,t}(z^t))} \pi(z'|z_t) \right\},$$

*for all $t \geq 0$, $z^t \in Z^t$, $z' \in Z$ and the initial conditions $a_{i,0} = 0$, for $i \in \mathbf{I}$, are an equilibrium with solvency constraints that are not too tight.*

**Proof:** See the appendix. ∎

However, the autarchic equilibrium is generally not (constrained) efficient, since the implied A-D prices may not satisfy condition 2.13. This result should not be surprising since, whenever some risk sharing is possible, all individuals may be better off by trading in the asset market, instead of consuming in autarky.

Although we can not guarantee that an A-J equilibrium is (constrained) efficient, under certain conditions, an A-J equilibrium allocation is the same as the corresponding K-L equilibrium allocation, which guarantees that the A-J allocation is constrained efficient. These sufficient conditions are stated in the following proposition.

**Proposition 2.4.5** *(Alvarez and Jermann (2000), page 784) Let $\{q, c_i, a_i\}$ be an A-J equilibrium given the solvency constraints $\{B_i\}$ and the initial wealth $a_{i,0} = 0$. Suppose (a) the implied interest rates are high, i.e., 2.13 holds, (b) the solvency constraints are not too tight, i.e. 2.10 holds, and (c) for each $i \in \mathbf{I}$ there is a constant $\xi_i$ such that for all $t$, $z^t$,*

$$|u(c_{i,t}(z^t))| \leq \xi_i \cdot u'(c_{i,t}(z^t)) \cdot c_{i,t}(z^t). \tag{2.14}$$

*Then, the consumption allocations $\{c_i\}$ and the A-D prices $\{Q_t\}$ derived from $\{q_t\}$ are a K-L equilibrium.*

**Proof:** See the appendix. ∎

**Remark 2.4.6** *(Alvarez and Jermann (2000), page 784) There are several instances in which the technical requirement 2.14 is satisfied, such as when $u(\cdot)$ has a RRA different from one at zero consumption, when $u'(0) < +\infty$, or when consumption is uniformly bounded away from zero.*

By theorem 2.4.5 and by the fact that every K-L equilibrium allocation is constrained efficient, we have the following version of the *first welfare theorem* for an A-J economy:

**Corollary 2.4.1** *(Alvarez and Jermann (2000), page 784) ($1^{rst}$ Welfare Theorem) If 2.14 is satisfied, an A-J equilibrium with solvency constraints that are not too tight (i.e., eq. 2.10 holds), and with high implied interest rates (i.e., eq. 2.13 holds), is constrained efficient.*

### 2.4.1 Further Results about A-J equilibrium

There is a class of parameters in the A-J model for which the first best allocation (i.e., the equilibrium allocation for an economy without solvency constraints) can be implemented by an A-J equilibrium. There is another class of parameters for which autarky is the only allocation that can be implemented by an A-J equilibrium (and therefore is the only feasible allocation). Both of these cases are not very interesting, since we would expect that the limited commitment would allow individuals to have less risk sharing than in an economy where there is perfect commitment (i.e., where there is no solvency constraints), but would allow all agents to be strictly better off than autarky by sharing some of their risks trough the asset market.

For this reason, A-J (2000) derives conditions under which the autarky is the only feasible allocation. They also show the (unsurprising) result that, if there is an efficient allocation other than autarky (i.e., an allocation that allows some risk sharing), then this allocation has high implied interest rates, and therefore, by the second welfare theorem, can be implemented by an A-J equilibrium with solvency constraints that are not too tight. We are now going to present these results.

By proposition 2.4.4 and the first welfare theorem (corollary 2.4.1) we have the following proposition:

**Proposition 2.4.7** *(Alvarez and Jermann (2000), page 785) If the implied interest rates for the autarky allocations $\{c_i = e_i\}$, are high, i.e., satisfy 2.13, then autarky is a constrained efficient allocation, and hence is the only feasible allocation.*

**Proof:** See the appendix. ■

The next proposition presents some sufficient conditions under which autarky is the only feasible allocation (or equivalently, the only equilibrium):

**Proposition 2.4.8** *(Alvarez and Jermann (2000), page 785) Autarky is the only feasible allocation in any of the following cases: (i) the time discount factor $\beta$ is sufficiently small, (ii) risk aversion is sufficiently small uniformly, (iii) the variance of the idiosyncratic shock is sufficiently close to zero, and (iv) the transition probability matrix of the stochastic process $z$ is sufficiently close to identity.*

**Proof:** See the appendix. ■

This result is intuitive because:

(i) If the discount factor is too small, then everyone will have high incentives to default. Therefore, the solvency constraints must be very tight to prevent individuals from defaulting. In the limit ($\beta \downarrow 0$), the credit limit must be zero to prevent default, so that autarky is the only feasible option.

(ii) If risk aversion is sufficiently small, individuals have less desire to smooth consumption, which makes autarky a more attractive option.

(iii) If the variance of each agent's endowment is sufficiently small, then the autarky allocation will be doing a good job in smoothing consumption.

(iv) If the shocks are very persistent, individual's wealth will not change very frequently, so that autarky will be doing a good job in smoothing consumption.

Finally we state the proposition according to which, whenever an efficient allocation allows some risk sharing, then it has high implied interest rates:

**Proposition 2.4.9** *(Alvarez and Jermann (2000), page 786) Let $\{c_i\}$ be a constrained efficient allocation. Assume that some risk sharing is possible, i.e., for all agents $i$ there is a $z^t$ such that*

$$\pi(z^t|z_0)[U(c_i)(z^t) - U(e_i)(z^t)] > 0. \tag{2.15}$$

*Then the implied interest rates are high (i.e., 2.13 holds).*

It follows from proposition 2.4.9 and from the second welfare theorem that, whenever there is an efficient allocation other than autarky, then it can be implemented by an A-J equilibrium with solvency constraints that are not too tight.

### 2.4.2 Implications for asset pricing

Since the asset structure of this economy is complete, we can price any complex security that involves payment in different periods and states. It is straightforward to show that the A-J equilibrium prices of any complex security is no smaller than the corresponding security prices of an A-D economy with no solvency constraints. This happens because A-J security prices are given by the sum of the maximum of agents' MRS agents, whereas the corresponding security in the standard A-D economy is given by the maximum of the sum of agents' MRS.

Intuitively, this happens because in an A-J economy agents sell less securities than they would be willing to, compared to an otherwise economy without solvency constraints. By market clearing, this implies that high-income individuals have to buy less securities. To induce high-income individuals to buy less securities, prices must increase.

One particular consequence of this result is that the interest rate of an A-J economy is smaller compared to an otherwise economy without solvency constraints. This result follows immediately from the fact that the interest rate equals the price of a state independent one period security.

### 2.4.3 Computational results (AJ-2001)

It may be computational difficult to compute equilibrium asset prices and allocations simultaneously. However, since this model has exogenous participation constraints and since there is an equivalence between constrained efficient allocations and A-J equilibrium allocations (first and second welfare theorems), we can first iteratively solve a planner's constrained first best problem (centralized problem) to find the equilibrium consumption allocation, and then substitute these allocations into the FOC's of the A-J problem to find the equilibrium asset prices and quantities of the decentralized problem.

In the graphs below we replicate the equilibrium allocations for the first example in A-J (2001), with two consumers, two states of nature and no aggregate growth. In this example, however, there is no need to iterate the value function of the constrained first best problem to find the equilibrium consumption allocation, since A-J (2001) fully characterize the equilibrium consumption for this example by a system of four equations. We used the bissection method to find the solution for this system.

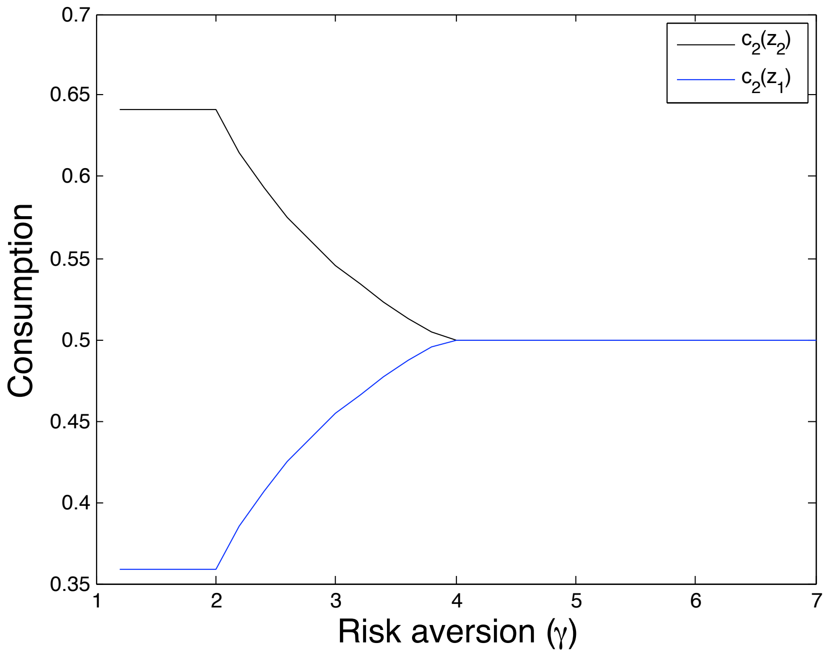


Figure 2.1: Optimal consumption allocations.

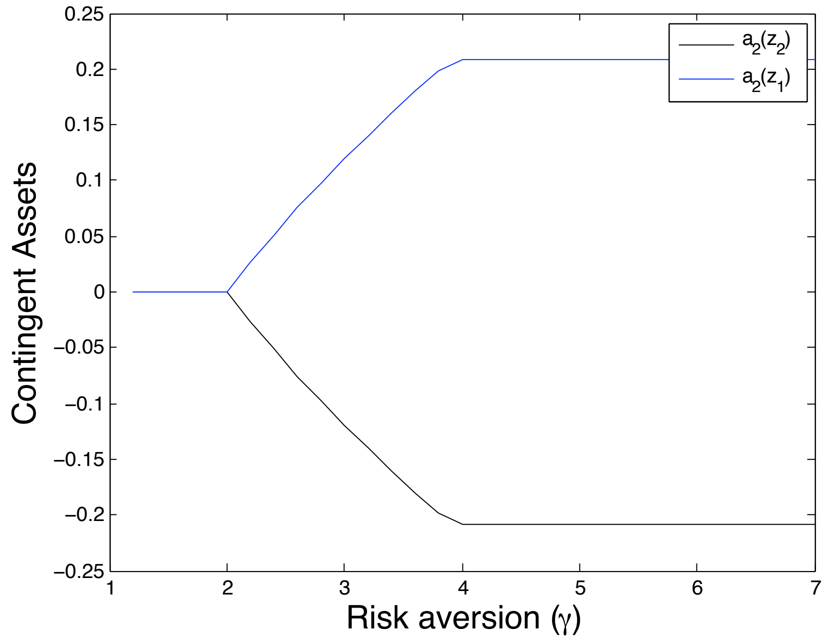


Figure 2.2: Contingent assets equilibrium.

# Appendix A

## A.1 Aggregate Growth in A-J

Suppose aggregate endowment $\{e_t(\cdot)\}$ follows the process

$$e_{t+1}(z^t, z_{t+1}) = e_t(z^t)\lambda(z_{t+1})$$
$$e_0(z_0) = 1$$

and that agent's $i$ endowment $\{e_{i,t}(\cdot)\}$ is given by

$$e_{i,t}(z^t) = e_t(z^t) \cdot \varepsilon_i(z_t)$$

Assume that utility is given by

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}.$$

Define

$$\hat{c}_{i,t}(z^t) \equiv \frac{c_{i,t}(z^t)}{e_t(z^t)} \qquad \hat{\pi}(z'|z) \equiv \frac{\pi(z'|z)\lambda(z')^{1-\gamma}}{\sum_z' \pi(z'|z)\lambda(z')^{1-\gamma}}$$
$$\hat{\beta}(z) \equiv \beta \sum_{z'} \pi(z'|z)\lambda(z')^{1-\gamma}.$$

Then the utility of consuming $\{\hat{c}_i\}$ with constant aggregate endowment 1 and contingent discount factor $\hat{\beta}(\cdot)$ is given by

$$\begin{aligned}
\hat{U}(\hat{c})(z_0) &= \sum_{t=0} \sum_{z^t \in Z^t} \left( \prod_{z_t \in (z^t)} \hat{\beta(z_t)}^t \right) \hat{c}_{i,t}(z^t)^{1-\gamma} \hat{\pi}(z_t|z_{t-1}) \\
&= \sum_{t=0} \sum_{z^t \in Z^t} \beta^t \left( \prod_{z_t \in (z^t)} \sum_{z_t} \pi(z_t|z_{t-1})\lambda(z_t)^{1-\gamma} \right) \frac{c_{i,t}(z^t)^{1-\gamma}}{e(z_0) \prod_{z_t \in (z^t)} \lambda(z_t)^{1-\gamma}} \frac{\prod_{z_t \in (z^t)} \pi(z_t|z_{t-1}) \prod_{z_t \in (z^t)} \lambda(z_t)^{1-\gamma}}{\prod_{z_t \in (z^t)} \sum_{z_t} \pi(z_t|z_{t-1})\lambda(z_t)^{1-\gamma}} \\
&= \sum_{t=0} \sum_{z^t \in Z^t} \beta^t c_{i,t}(z^t)^{1-\gamma} \prod_{z_t \in (z^t)} \pi(z_t|z_{t-1}) \\
&= \sum_{t=0} \sum_{z^t \in Z^t} \beta^t c_{i,t}(z^t)^{1-\gamma} \pi(z^t|z_0) \equiv U(c)(z_0),
\end{aligned}$$

which is the utility of consuming $\{c_i\}$, when the discount factor is $\beta$, aggregate growth is $\lambda(\cdot)$ and the probabilities are $\pi$. Moreover, it can be easily shown that resource feasibility and participation constraints are satisfied for an allocation $\{\hat{c}_i\}_{i\in\mathbf{I}}$ in an economy with aggregate wealth constant and equal to 1, discount factor $\hat{\beta}(\cdot)$ and probabilities $\hat{\pi}$ *if and only if* they are satisfied for the corresponding $\{c_i\}_{i\in\mathbf{I}}$ allocation, with aggregate growth $\lambda(\cdot)$, constant discount factor $\beta$ and probabilities $\pi$.

## A.2 Proofs

**Proof of proposition 2.3.1:** The Lagrangian associated with consumer's $i$ problem in the K-L equilibrium can be written as

$$L(\{c_i\},\zeta_i,\{\eta_i\}) = U(c_i)(z_0) + \zeta_i\left[\sum_{t\geq 0}\sum_{z^t\in Z^t}(e_{i,t}(z^t)-c_{i,t}(z^t))Q_0(z^t|z_0)\right] + \sum_{t\geq 0}\sum_{z^t\in Z^t}\beta^t\eta_{i,t}(z^t)\left(U(e_i)(z^t)-U(c_i)(z^t)\right)\pi(z^t|z_0).$$

By the FOC we have that

$$[c_{i,t+1}(z^{t+1})]:\quad \beta^{t+1}u'(c_{i,t+1}(z^{t+1}))\pi(z^{t+1}|z_0) - \zeta_i Q_0(z^{t+1}|z_0) + \sum_{z^r\preceq z^{t+1}}\eta_{i,r}(z^r)\beta^{t+1}u'(c_{i,t+1}(z^{t+1}))\pi(z^{t+1}|z_0) = 0$$

$$\Rightarrow \beta^{t+1}u'(c_{i,t+1}(z^{t+1}))\pi(z^{t+1}|z_0)\left[1+\sum_{z^r\preceq z^{t+1}}\eta_{i,r}(z^r)\right] = \zeta_i Q_0(z^{t+1}|z_0) \qquad \text{(A.1)}$$

and, for all $z^t \preceq z^{t+1}$,

$$[c_{i,t}(z^t)]:\quad \beta^t u'(c_{i,t}(z^t))\pi(z^t|z_0) - \zeta_i Q_0(z^t|z_0) + \sum_{z^r\preceq z^t}\eta_{i,r}(z^r)\beta^t u'(c_{i,t}(z^t))\pi(z^t|z_0) = 0$$

$$\Rightarrow \beta^t u'(c_{i,t}(z^t))\pi(z^t|z_0)\left[1+\sum_{z^r\preceq z^t}\eta_{i,r}(z^r)\right] = \zeta_i Q_0(z^t|z_0) \qquad \text{(A.2)}$$

Dividing A.1 by A.2 we have that

$$q_{0,t}(z^{t+1}) = \frac{Q_0(z^{t+1}|z_0)}{Q_0(z^t|z_0)} = \frac{\beta^{t+1}u'(c_{i,t+1}(z^{t+1}))\pi(z^{t+1}|z_0)\left[1+\sum_{z^r\preceq z^{t+1}}\eta_{i,r}(z^r)\right]}{\beta^t u'(c_{i,t}(z^t))\pi(z^t|z_0)\left[1+\sum_{z^r\preceq z^t}\eta_{i,r}(z^r)\right]} = \beta\frac{u'(c_{i,t+1}(z^{t+1}))\left[1+\sum_{z^r\preceq z^{t+1}}\eta_{i,r}(z^r)\right]}{u'(c_{i,t}(z^t))\left[1+\sum_{z^r\preceq z^t}\eta_{i,r}(z^r)\right]}\pi(z^{t+1}|z_t) \qquad \text{(A.3)}$$

But if $U(c_i)(z^{t+1}) > U(e_i)(z^{t+1})$, then

$$\eta_{i,t}(z^{t+1}) = 0$$
$$\Rightarrow \sum_{z^r\preceq z^{t+1}}\eta_{i,r}(z^r) = \sum_{z^r\preceq z^t}\eta_{i,r}(z^r) \qquad \text{(A.4)}$$

Substituting A.4 into A.3, we have that, if $U(c_i)(z^{t+1}) > U(e_i)(z^{t+1})$, then

$$q_{0,t}(z^{t+1}) = \beta \frac{u'(c_{i,t+1}(z^{t+1}))}{u'(c_{i,t}(z^t))} \pi(z^{t+1}|z_t),$$

which proves the second part of the proposition.

Now notice that, since

$$\frac{\left[1 + \sum_{z^r \preceq z^{t+1}} \eta_{i,r}(z^r)\right]}{\left[1 + \sum_{z^r \preceq z^t} \eta_{i,r}(z^r)\right]} \geq 1,$$

we have that, $\forall i \in \mathbf{I}$,

$$q_{0,t}(z^{t+1}) = \beta \frac{u'(c_{i,t+1}(z^{t+1})) \left[1 + \sum_{z^r \preceq z^{t+1}} \eta_{i,r}(z^r)\right]}{u'(c_{i,t}(z^t)) \left[1 + \sum_{z^r \preceq z^t} \eta_{i,r}(z^r)\right]} \pi(z^{t+1}|z_t) \geq \beta \frac{u'(c_{i,t+1}(z^{t+1}))}{u'(c_{i,t}(z^t))} \pi(z^{t+1}|z_t)$$

$$\Longleftrightarrow \quad q_{0,t}(z^{t+1}) \geq \beta \frac{u'(c_{i,t+1}(z^{t+1}))}{u'(c_{i,t}(z^t))} \pi(z^{t+1}|z_t)$$

Thus, $q_{0,t}(z^{t+1})$ is an upper bound for $\beta \frac{u'(c_{i,t+1}(z^{t+1}))}{u'(c_{i,t}(z^t))} \pi(z^{t+1}|z_t)$. Since this upper bound is achieved by an unconstrained agent, we have proved the first part of the proposition whenever there is at least one unconstrained agent.[1] ■

**Proof of proposition 2.4.1:**

Let $\{c_i^*, a_i^*\}$ be a feasible allocation that satisfies 2.8 and 2.9 and let $\{c_i^*, a_i\}$ be any feasible allocation. Define

$$\begin{aligned}
D &\equiv \lim_{T\to\infty} \sum_{t=0}^{T} \sum_{z^t \in Z^t} \beta^t \left[u(c_{i,t}^*(z^t)) - u(c_{i,t}(z^{t+1}))\right] \pi(z^t|z_0) \\
&= \lim_{T\to\infty} \sum_{t=0}^{T} \sum_{z^t \in Z^t} \beta^t \Bigg[ u \left( e_{i,t} + a_{i,t}^*(z^t) - \sum_{z^{t+1} \in Z^{t+1}} a_{i,t+1}^*(z^{t+1}) q_t(z^{t+1}) \right) - \\
&\quad - u \left( e_{i,t} + a_{i,t}(z^t) - \sum_{z^{t+1} \in Z^{t+1}} a_{i,t+1}(z^{t+1}) q_t(z^{t+1}) \right) \Bigg] \pi(z^t|z_0) \qquad \text{(A.5)}
\end{aligned}$$

Since $u$ is concave, continuous and differentiable and since the feasible set is convex, we have that

$$\begin{aligned}
D \geq \lim_{T\to\infty} \sum_{t=0}^{T} \sum_{z^t \in Z^t} \beta^t \Big\{ & u'(c_{i,t}^*(z^t)) \left[a_{i,t}^*(z^t) - a_{i,t}(z^t)\right] \\
& - q_t(z^{t+1}) u'(c_{i,t}(z^{t+1})) \left[a_{i,t+1}^*(z^{t+1}) - a_{i,t+1}(z^{t+1})\right] \Big\} \pi(z^t|z_0)
\end{aligned}$$

[1] Using the equivalence between the not too tight solvency constraints and the participation constraints being binding, and using the fact that agents who lend in the A-J equilibrium are never constrained ($B_{i,t} \leq 0$), we have that, whenever there is some asset trade in period $t$, then there is at least one unconstrained agent at $t$ (because at least one agent will be lending money at this period).

Since $a_{i,0}(z_0) = a_{i,0}(z_0) = 0 \quad \forall z_0 \in Z$, rearranging terms gives

$$D \geq \lim_{T\to\infty} \left\{ \sum_{t=0}^{T-1} \sum_{z^t \in Z^t} \beta^t \left[ \beta u'(c^*_{i,t+1}(z^{t+1}))\pi(z^{t+1}|z^t) - q_t(z^{t+1})u'(c_{i,t}(z^t)) \right] \right.$$
$$\cdot \left[ a^*_{i,t+1}(z^{t+1}) - a_{i,t+1}(z^{t+1}) \right] \pi(z^t|z_0)$$
$$\left. - \sum_{z^T \in Z^T} \beta^T q_T(z^{T+1})u'(c_{i,T}(z^T)) \left[ a^*_{i,T+1}(z^{T+1}) - a_{i,T+1}(z^{T+1}) \right] \pi(z^T|z_0) \right\}$$

Now notice that the terms in the summation are all greater than zero. Indeed, if $a^*_{i,t+1}(z^{t+1}) > B_{i,t+1}(z^{t+1})$, then, by the Euler equation, the corresponding term of the summation is zero. Suppose, on the other hand, that

$$a^*_{i,t+1}(z^{t+1}) = B_{i,t+1}(z^{t+1}).$$

Then, since

$$a_{i,t+1}(z^{t+1}) \geq B_{i,t+1}(z^{t+1}) = a^*_{i,t+1}(z^{t+1}),$$

we have that

$$\left( a^*_{i,t+1}(z^{t+1}) - a_{i,t+1}(z^{t+1}) \right) \leq 0. \tag{A.6}$$

But, by the Euler equation we have that

$$\beta u'(c^*_{i,t+1}(z^{t+1}))\pi(z^{t+1}|z^t) - q_t(z^{t+1})u'(c_{i,t}(z^t)) \leq 0,$$

which, together with A.6, implies that the corresponding term of the summation is greater or equal than zero.

Therefore,

$$D \geq - \lim_{T\to\infty} \sum_{z^T \in Z^T} \beta^T q_T(z^{T+1})u'(c_{i,T}(z^T)) \left[ a^*_{i,T+1}(z^{T+1}) - a_{i,T+1}(z^{T+1}) \right] \pi(z^T|z_0)$$
$$\geq - \lim_{T\to\infty} \sum_{z^T \in Z^T} \beta^T q_T(z^{T+1})u'(c_{i,T}(z^T)) \left[ a^*_{i,T+1}(z^{T+1}) - B_{i,T+1}(z^{T+1}) \right] \pi(z^T|z_0),$$

where the second inequality follows from the fact that $\beta^T q_T(z^{T+1})u'(c_{i,T}(z^T))$ is strictly positive and that $a_{i,T+1}(z^{T+1}) \geq B_{i,T+1}(z^{T+1})$.

Using the Euler equation again, this implies that

$$D \geq \lim_{T\to\infty} \sum_{z^T \in Z^T} \beta^{T+1} u'(c_{i,T+1}(z^T)) \left[ a^*_{i,T+1}(z^{T+1}) - B_{i,T+1}(z^{T+1}) \right] \pi(z^{T+1}|z_0) \overset{(*)}{=} 0$$

where (*) follows from the transversality condition. ■

**Proof of proposition 2.4.2:**

If $\{c_i, a_i\}$ is an A-J equilibrium allocation with solvency constraints $\{B_i\}$ that are not too tight, then it is immediate that

$$U(c_i)(z^t) \geq U(e_i)(z^t) \quad \text{and}$$
$$U(c_i)(z^t) = U(e_i)(z^t) \Leftarrow a_{i,t}(z^t) = B_{i,t}(z^t).$$

Now, since

$$J_{i,t}(a_i(z^t), z^t) = U(c_i)(z^t) \geq U(e_i)(z^t),$$

if we use the fact that $J_{i,t}(\cdot, z^t)$ is strictly increasing, we have that $a_{i,t}(z^t) > B_{i,t}(z^t)$ implies that

$$J_{i,t}(a_i(z^t), z^t) = U(c_i)(z^t) > U(e_i)(z^t).$$

Therefore,

$$U(c_i)(z^t) = U(e_i)(z^t) \Rightarrow a_{i,t}(z^t) = B_{i,t}(z^t).$$

■

**Proof of proposition 2.4.4:**

We only need to show that the allocation and prices

$$c_{a,i,t}(z^t) = e_{i,t}(z^t), \quad a_{a,i,t+1}(z^t, z') \equiv B_{a,i,t}(z^t) = 0,$$
$$q_{a,t}(z^t, z') \equiv \max_{i \in \mathbf{I}} \left\{ \beta \frac{u'(e_{i,t+1}(z^t, z'))}{u'(e_{i,t}(z^t))} \pi(z'|z_t) \right\},$$

satisfy the Euler equations (eq. 2.8), the transversality condition (eq. 2.9), the resource feasible constraint and the asset market clearing condition. It is obvious that this allocation satisfies the transversality condition (because the term $[a_{i,t}(z^t) - B_{i,t}(z^t)]$ in the transversality condition is always zero), the resource feasible constraint and the asset market clearing condition. Let us then prove that the Euler equations are satisfied. First notice that

$$q_{a,t}(z^t, z') = \max_{i \in \mathbf{I}} \left\{ \beta \frac{u'(e_{i,t+1}(z^t, z'))}{u'(e_{i,t}(z^t))} \pi(z'|z_t) \right\}$$
$$\Rightarrow q_{a,t}(z^t, z') \geq \beta \frac{u'(e_{i,t+1}(z^t, z'))}{u'(e_{i,t}(z^t))} \pi(z'|z_t) \quad \forall i \in \mathbf{I}$$
$$\iff -u'(e_{i,t}(z^t)) q_{a,t}(z^t, z') + \beta u'(e_{i,t+1}(z^t, z')) \pi(z'|z_t) \leq 0,$$

which proves that the Euler condition is satisfied. ■

**Proof of proposition 2.4.5**

Let $\{q, c_i, a_i\}$ be an A-J equilibrium with solvency constraints that are not too tight. Then the consumption allocations are resource feasible and satisfy the participation constraint for each agent. The budget constraint is also satisfied, since 2.6

implies that

$$
\begin{aligned}
&e_{i,t}(z^t) + a_{i,t}(z^t) = \sum_{z^{t+1}\succeq z^t} a_{i,t+1}(z^{t+1})q_t(z^{t+1}) + c_{i,t}(z^t) \\
&\Rightarrow \sum_{t=0}\sum_{z^t\in Z^t} Q_0(z^t|z_0)c_{i,t}(z^t) = \sum_{t=0}\sum_{z^t\in Z^t} Q_0(z^t|z_0)e_{i,t}(z^t) + \sum_{t=0}\sum_{z^t\in Z^t} Q_0(z^t|z_0)a_{i,t}(z^t) \\
&-\sum_{t=0}\sum_{z^{t+1}\in Z^{t+1}} Q_0(z^{t+1}|z_0)c_{i,t}(z^{t+1}) \\
&\iff \sum_{t=0}\sum_{z^t\in Z^t} Q_0(z^t|z_0)c_{i,t}(z^t) = \sum_{t=0}\sum_{z^t\in Z^t} Q_0(z^t|z_0)e_{i,t}(z^t) + \sum_{t=1}\sum_{z^t\in Z^t} Q_0(z^t|z_0)a_{i,t}(z^t) \\
&-\sum_{t=1}\sum_{z^t\in Z^t} Q_0(z^t|z_0)c_{i,t}(z^t) \\
&\iff \sum_{t=0}\sum_{z^t\in Z^t} Q_0(z^t|z_0)c_{i,t}(z^t) = \sum_{t=0}\sum_{z^t\in Z^t} Q_0(z^t|z_0)e_{i,t}(z^t),
\end{aligned}
$$

where the second inequality follows from $a_{i,0}(z_0) = 0 \quad \forall z_0 \in Z$.

Therefore, It only remains to show that, given the A-D prices $\{Q_0\}$ implied by $\{q\}$, $\{c_i\}$ solves

$$
\begin{aligned}
&\max_{\{\bar{c}_i\}} U(\bar{c}_i)(z_0), \\
s.t.& \\
&\sum_{t\geq 0}\sum_{z^t\in Z^t} \bar{c}_{i,t}(z^t)Q_0(z^t|z_0) \leq \sum_{t\geq 0}\sum_{z^t\in Z^t} e_{i,t}(z^t)Q_0(z^t|z_0) \\
&U(\bar{c}_i)(z^t) \geq U(e_i)(z^t) \quad \forall t, \forall z^t.
\end{aligned}
$$

As before, we can write the Lagrangian associated to this problem as

$$
\begin{aligned}
L(\{\bar{c}_i\}, \zeta_i, \{\eta_i\}) = U(\bar{c}_i)(z_0) + \zeta_i \left[\sum_{t\geq 0}\sum_{z^t\in Z^t} (e_{i,t}(z^t) - \bar{c}_{i,t}(z^t))Q_0(z^t|z_0)\right] \\
+ \sum_{t\geq 0}\sum_{z^t\in Z^t} \beta^t \eta_{i,t}(z^t)\left(U(e_i)(z^t) - U(\bar{c}_i)(z^t)\right)\pi(z^t|z_0).
\end{aligned}
$$

Therefore, by the *saddle point theorem*, we only need to find positive multipliers $(\zeta_i, \{\eta_i\})$ such that, given $\{c_i\}$, $(\zeta_i, \{\eta_i\})$ minimizes $L(\{\bar{c}_i\}, \cdot, \cdot)$, and, given $(\zeta_i, \{\eta_i\})$, $\{c_i\}$ maximizes $L(\cdot, \zeta_i, \{\eta_i\})$.

Our guess for the multipliers are

$$
\zeta_i = \frac{u'(c_{i,0}(z_0))}{Q)(z_0|z_0)} = u'(c_{i,0}(z_0))
$$

and $\{\eta_i\}$ such that for all $t$, $z^t$,

$$
\Rightarrow \beta^t u'(c_{i,t}(z^t))\pi(z^t|z_0)\left[1 + \sum_{z^r\preceq z^t} \eta_{i,r}(z^r)\right] = \zeta_i Q_0(z^t|z_0) \qquad \text{(A.7)}
$$

Since the budget constraint must be valid with equality, it is immediate that $\zeta_i = u'(c_{i,0}(z_0)) > 0$ minimizes the Lagrangian given the allocation $\{c_i\}$. Let us then turn our attention to the $\{\eta_i\}$ multipliers and show that they also minimize the Lagrangian given the allocation $\{c_i\}$.

By adding one period to equation A.7, we have that for any $z^{t+1} \succeq z^t$,

$$\Rightarrow \beta^{t+1} u'(c_{i,t+1}(z^{t+1}))\pi(z^{t+1}|z_0)\left[1 + \sum_{z^r \preceq z^{t+1}} \eta_{i,r}(z^r)\right] = \zeta_i Q_0(z^{t+1}|z_0). \tag{A.8}$$

Dividing A.8 by A.7, we obtain

$$\begin{aligned} &q_{0,t}(z^{t+1}) = \beta \frac{u'(c_{i,t+1}(z^{t+1}))\left[1 + \sum_{z^r \preceq z^{t+1}} \eta_{i,r}(z^r)\right]}{u'(c_{i,t}(z^t))\left[1 + \sum_{z^r \preceq z^t} \eta_{i,r}(z^r)\right]} \pi(z^{t+1}|z_t) \\ &\Rightarrow -u'(c_{i,t}(z^t))q_t(z^t, z') + \beta\pi(z', z)u'(c_{i,t+1}(z^t, z'))\frac{\left[1 + \sum_{z^r \preceq z^{t+1}} \eta_{i,r}(z^r)\right]}{\left[1 + \sum_{z^r \preceq z^t} \eta_{i,r}(z^r)\right]} = 0 \\ &\Rightarrow -u'(c_{i,t}(z^t))q_t(z^t, z') + \beta\pi(z', z)u'(c_{i,t+1}(z^t, z')) \leq 0. \end{aligned} \tag{A.9}$$

But by the Euler equation, whenever $U(c_i)(z^{t+1}) > U(e_i)(z^{t+1})$, then A.9 must be valid with equality, which happens iff

$$\begin{aligned} \frac{\left[1 + \sum_{z^r \preceq z^{t+1}} \eta_{i,r}(z^r)\right]}{\left[1 + \sum_{z^r \preceq z^t} \eta_{i,r}(z^r)\right]} &= 1 \\ \iff \eta_{i,t}(z^{t+1}) &= 0. \end{aligned}$$

Therefore, we have that in fact the multipliers $\{\eta_i\}$ minimize the Lagrangian for the fixed allocation $\{c_i\}$.

Now we only need to show that $\{c_i\}$ maximizes $L(\cdot, \zeta_i, \{\eta_i\})$, which is equivalent to show that[2]

$$
\begin{aligned}
&U(c_i)(z_0) + \zeta_i \left[ \sum_{t\geq 0} \sum_{z^t \in Z^t} (e_{i,t}(z^t) - c_{i,t}(z^t)) Q_0(z^t|z_0) \right] \\
&\quad + \sum_{t\geq 0} \sum_{z^t \in Z^t} \beta^t \eta_{i,t}(z^t) \left( U(e_i)(z^t) - U(c_i)(z^t) \right) \pi(z^t|z_0) \\
\geq &U(\overline{c}_i)(z_0) + \zeta_i \left[ \sum_{t\geq 0} \sum_{z^t \in Z^t} (e_{i,t}(z^t) - \overline{c}_{i,t}(z^t)) Q_0(z^t|z_0) \right] \\
&\quad + \sum_{t\geq 0} \sum_{z^t \in Z^t} \beta^t \eta_{i,t}(z^t) \left( U(e_i)(z^t) - U(\overline{c}_i)(z^t) \right) \pi(z^t|z_0) \qquad \text{(A.10)}
\end{aligned}
$$

$\Longleftrightarrow$

$$
\begin{aligned}
&U(c_i)(z_0) - \zeta_i \left[ \sum_{t\geq 0} \sum_{z^t \in Z^t} Q_0(z^t|z_0) c_{i,t}(z^t) \right] + \sum_{t\geq 0} \sum_{z^t \in Z^t} \beta^t \eta_{i,t}(z^t) U(c_i)(z^t) \pi(z^t|z_0) \\
&\geq U(\overline{c}_i)(z_0) - \zeta_i \left[ \sum_{t\geq 0} \sum_{z^t \in Z^t} Q_0(z^t|z_0) \overline{c}_{i,t}(z^t) \right] + \sum_{t\geq 0} \sum_{z^t \in Z^t} \beta^t \eta_{i,t}(z^t) U(\overline{c}_i)(z^t) \pi(z^t|z_0) \\
&\qquad \text{(A.11)}
\end{aligned}
$$

for any $\{\overline{c}_i\}$ that satisfy the budget and the participation constraints.

Rearranging the terms in inequality A.11, our necessary and sufficient condition for optimum becomes

$$
\begin{aligned}
&\sum_{z^t \succeq Z^t} \beta^t u(c_{i,t}(z^t)) \left[ 1 + \sum_{z^r \preceq z^t} \eta_{i,t}(z^t) \right] \pi(z^t|z_0) - \zeta_i \left[ \sum_{z^t \succeq z_0} Q_0(z^t|z_0) c_{i,t}(z^t) \right] \\
&\geq \sum_{z^t \succeq Z^t} \beta^t u(\overline{c}_{i,t}(z^t)) \left[ 1 + \sum_{z^r \preceq z^t} \eta_{i,t}(z^t) \right] \pi(z^t|z_0) - \zeta_i \left[ \sum_{z^t \succeq z_0} Q_0(z^t|z_0) \overline{c}_{i,t}(z^t) \right] \\
&\qquad \text{(A.12)}
\end{aligned}
$$

Since $u$ is concave and differentiable, we have that

$$
u(\overline{c}_{i,t}(z^t)) \leq u(c_{i,t}(z^t)) + u'(c_{i,t}(z^t))[\overline{c}_{i,t}(z^t) - c_{i,t}(z^t)] \qquad \text{(A.13)}
$$

[2]The conditions 2.13 and 2.14 guarantee that the left hand side of equations A.10 is finite. This implies that the optimal consumption condition stated in this inequality is correct (see Alvarez an Jermann (2000)).

Using this inequality we have the desired result:

$$\sum_{z^t \succeq Z^t} \beta^t u(\overline{c}_{i,t}(z^t)) \left[1 + \sum_{z^r \preceq z^t} \eta_{i,t}(z^t)\right] \pi(z^t|z_0) - \zeta_i \left[\sum_{z^t \succeq z_0} Q_0(z^t|z_0)\overline{c}_{i,t}(z^t)\right]$$

$$\leq \sum_{z^t \succeq Z^t} \beta^t u(c_{i,t}(z^t)) \left[1 + \sum_{z^r \preceq z^t} \eta_{i,t}(z^t)\right] \pi(z^t|z_0)$$

$$+ \sum_{z^t \succeq Z^t} \beta^t u'(c_{i,t}(z^t))[\overline{c}_{i,t} - c_{i,t}(z^t)] \left[1 + \sum_{z^r \preceq z^t} \eta_{i,t}(z^t)\right] \pi(z^t|z_0)$$

$$- \zeta_i \left[\sum_{z^t \succeq z_0} Q_0(z^t|z_0)\overline{c}_{i,t}(z^t)\right]$$

$$= \sum_{z^t \succeq Z^t} \beta^t u(c_{i,t}(z^t)) \left[1 + \sum_{z^r \preceq z^t} \eta_{i,t}(z^t)\right] \pi(z^t|z_0) - \zeta_i \left[\sum_{z^t \succeq z_0} Q_0(z^t|z_0)c_{i,t}(z^t)\right],$$

where the first inequality follows directly from A.13 and the last equality follows from the definition of $\{\eta_i\}$ (eq. A.7). ■

**Proof of proposition 2.4.7:** By proposition 2.4.4, $\{e_i, a_i = 0\}$ and the prices implied by this allocation are an A-J equilibrium with solvency constraints that are not too tight. Then, if the implied interest rates of the autarkic allocation $\{c_i = e_i\}$ are high, by theorem 2.4.1, this allocation can be implemented by a K-L equilibrium and, therefore, is constrained efficient, provided condition 2.14 is satisfied.

Now let's show that an autarkic allocation is an A-J equilibrium if and only if it is the only feasible allocation. If the only feasible allocation is the autarkic one, then it is immediate that the only A-J equilibrium will be the autarkic one. Now suppose the only A-J equilibrium is the autarkic one, then $B_{i,t}(z^t) = 0 \; \forall \; i, \, t, \, z^t$, which implies that every agent can never borrow, which implies that the only feasible allocation is the autarkic one. ■

**Proof of proposition 2.4.8:** To prove this proposition, we only need to show that, as conditions (i) to (iv) are satisfied, autarky has high implied interest rates.

First, notice that, since Arrow prices of the autarkic allocation,

$$q_t(z^{t+1}) = \beta \max_{i \in \mathbf{I}} \left\{ \frac{u'(e_i(z_{t+1}))}{u'(e_i(z_t))} \pi(z_{t+1}|z_t) \right\},$$

only depends on $z_t$ and $z_{t+1}$, we can write the autarkic Arrow prices as $q_a(z_t, z_{t+1})$.

Let us define the value of endowments evaluated at the autarkic prices $q_a(z_t, z_{t+1})$, when the current shock is $z_t$, as $A(z_t)$. Then, the following recursion must be satisfied:

$$A(z_t) = \sum_{z_{t+1} \in Z} q_a(z_t, z_{t+1}) \left[e(z_{t+1} + A(z_{t+1}))\right] \pi(z_t|z_{t+1}). \qquad \text{(A.14)}$$

Defining

$$\beta^*(z^t) \equiv \beta \sum_{z_{t+1}\in Z} \max_{i\in\mathbf{I}} \left\{ \frac{u'(e(z_{t+1}))}{u'(e(z_t))} \right\} \pi(z_{t+1}|z_t)$$

and

$$\pi^*(z_{t+1}|z_t) \equiv \max_{i\in\mathbf{I}} \left\{ \frac{u'(e(z_{t+1}))}{u'(e(z_t))} \right\} \pi(z_{t+1}|z_t) \Big/ \sum_{z_{t+1}\in Z} \max_{i\in\mathbf{I}} \left\{ \frac{u'(e(z_{t+1}))}{u'(e(z_t))} \right\} \pi(z_{t+1}|z_t),$$

equation A.14 can be rewritten as

$$A(z_{t+1}) = \beta^*(z_t) \sum_{z_{t+1}\in Z} \left[e(z_{t+1}) + A(z_{t+1})\right] \pi^*(z_{t+1}|z_t). \tag{A.15}$$

Let $S$ as the set of all real functions $f : Z \to \mathbb{R}$ defined on $Z$, with the sup norm, i.e., $||f|| = \max_{z\in Z} |f(z)|$. Then clearly, $S$ is a complete metric space with the sup norm (i.e., it is a Banach space with the sup norm). Now define the operator $T : S \to S$, such that, $\forall f \in S$ and $\forall z \in Z$,

$$T(f(z)) = \beta^*(z_t) \sum_{z_{t+1}\in Z} \left[e(z_{t+1}) + f(z_{t+1})\right] \pi^*(z_{t+1}|z_t).$$

Clearly, whenever $\beta^*(z) < 1\ \forall z \in Z$, this operator satisfies the Blackwell sufficient conditions for a contraction (discount and monotonicity). Therefore, by the contraction theorem, if $\beta^*(z) < 1\ \forall z \in Z$, $T$ must have a unique fixed point in $S$. Since $A : Z \to \mathbb{R}$ is a fixed point of the operator $T : S \to S$, we have that, whenever $\beta^*(z) < 1\ \forall z \in Z$, $A$ is bounded (i.e., whenever $\beta^*(z) < 1\ \forall z \in Z$, the autarch allocation has high implied interest rates).

Therefore, it only remains to show that, whenever (i), (ii), (iii) or (iv) are satisfied, $\beta^*(z) < 1\ \forall z \in Z$:

i) Since $e_i(z) > 0\ \forall z \in Z$, we have that $\sum_{z_{t+1}\in Z} \max_{i\in\mathbf{I}} \left\{ \frac{u'(e(z_{t+1}))}{u'(e(z_t))} \right\} \pi(z_{t+1}|z_t) < +\infty$. Therefore, $\exists\ \beta \in (0, 1)$ sufficiently small, such that

$$\beta^*(z^t) \equiv \beta \sum_{z_{t+1}\in Z} \max_{i\in\mathbf{I}} \left\{ \frac{u'(e(z_{t+1}))}{u'(e(z_t))} \right\} \pi(z_{t+1}|z_t) < 1.$$

ii) If the utility is of the CRRA form, i.e., if $u(c) = \frac{c^(1-\gamma)}{1-\gamma}$, then clearly

$$\lim_{\gamma\to 0} \frac{u'(e_i(z_{t+1}))}{u'(e_i(z_t))} = \lim_{\gamma\to 0} \left( \frac{e_i(z_{t+1})}{e_i(z_t)} \right)^{-\gamma} = 1 \qquad \forall z_t, z_{t+1} \in Z$$

$$\lim_{\gamma\to 0} \sum_{z_{t+1}\in Z} \max_{i\in\mathbf{I}} \left\{ \frac{u'(e_i(z_{t+1}))}{u'(e_i(z_t))} \right\} \pi(z_{t+1}|z_t) = 1 \quad \forall z_t \in Z.$$

$$\Rightarrow \lim_{\gamma\to 0} \beta^*(z^t) = \beta \lim_{\gamma\to 0} \sum_{z_{t+1}\in Z} \max_{i\in\mathbf{I}} \left\{ \frac{u'(e_i(z_{t+1}))}{u'(e_i(z_t))} \right\} \pi(z_{t+1}|z_t) = \beta < 1 \quad \forall z_t \in Z.$$

iii) For each $z \in Z$, create sequences of endowments for agent $i$ $\left\{e_i^j(z)\right\}_{j\in\mathbb{N}}$, such that for all $z \in Z$, $e_i^j(z) \to e_i > 0$, i.e., such that the variance of his idiosyncratic shocks converges to zero. Clearly, since $u : \mathbb{R}_+ \to \mathbb{R}$ is $C^1$, $g : \mathbb{R}_+^2 \to \mathbb{R}$ such that $g(x, y) = \frac{u'(x)}{u'(y)}$ is continuous. Moreover, $g(e_i, e_i) = \frac{u'(e_i)}{u'(e_i)} = 1$. Therefore, by continuity of $g(\cdot, \cdot)$, we have that

$$\lim_{j\to\infty} \frac{u'(e_i^j(z_{t+1}))}{u'(e_i^j(z_t))} = 1 \qquad \forall z_t, z_{t+1} \in Z$$

$$\Rightarrow \lim_{j\to\infty} \sum_{z_{t+1}\in Z} \max_{i\in\mathbf{I}} \left\{ \frac{u'(e_i^j(z_{t+1}))}{u'(e_i^j(z_t))} \right\} \pi(z_{t+1}|z_t) = 1 \quad \forall z_t \in Z$$

$$\Rightarrow \lim_{j\to\infty} \beta_j^*(z^t) \equiv \beta \lim_{j\to\infty} \sum_{z_{t+1}\in Z} \max_{i\in\mathbf{I}} \left\{ \frac{u'(e_i^j(z_{t+1}))}{u'(e_i^j(z_t))} \right\} \pi(z_{t+1}|z_t) = \beta < 1 \quad \forall z_t \in Z$$

iv) If the transition probability of shocks $\{\Pi_j\}$ converges to identity, then

$$\lim_{j\to\infty} \pi^j(z_{t+1}|z_t) = \begin{cases} 1, & \text{if } z_{t+1} = z_t \\ 0, & \text{otherwise.} \end{cases} \tag{A.16}$$

Moreover,

$$\frac{u'(e_i(z_{t+1}))}{u'(e_i(z_t))} = \begin{cases} 1, & \text{if } z_{t+1} = z_t \\ 0, & \text{otherwise} \end{cases}, \tag{A.17}$$

then, by A.16 and A.17 we clearly have that

$$\lim_{j\to\infty} \sum_{z_{t+1}\in Z} \max_{i\in\mathbf{I}} \left\{ \frac{u'(e_i(z_{t+1}))}{u'(e_i(z_t))} \right\} \pi^j(z_{t+1}|z_t) = 1 \quad \forall z_t \in Z$$

$$\Rightarrow \lim_{j\to\infty} \beta_j^*(z^t) \equiv \beta \lim_{j\to\infty} \sum_{z_{t+1}\in Z} \max_{i\in\mathbf{I}} \left\{ \frac{u'(e_i(z_{t+1}))}{u'(e_i(z_t))} \right\} \pi^j(z_{t+1}|z_t) = \beta < 1 \quad \forall z_t \in Z$$

■

# Chapter 3

# The effect of changes in the volatility of income on the volatility of consumption

## 3.1 Introduction

Dirk Krueger and Fabrizio Perri (2005) show that in Alvarez and Jermann (2000) and Kehoe and Levine (2001) environment, a change in the volatility of income may be followed by a smaller change in the volatility of consumption. Surprisingly, under certain conditions, an increase in the volatility of income may even reduce the volatility of consumption. The intuition for this result comes from the fact that, by allowing the volatility of income to increase, the outside option of defaulting becomes more expensive, which increases the level of commitment in the economy, thereby allowing more risk sharing. We are now going to present this result in a simple environment, with only two possible states of nature and two agents.

## 3.2 Environment

Consider the same environment described in section 2.2, except that now we assume that:

- There are only two agents in the economy: $\mathbf{I} = \{1, 2\}$
- There are only two states of nature: $z_t \in \{H, L\}$
- The endowments of the two agents alternate between a god and a bad state: Whenever $z_t = H$, agent 1 has endowment $e_t = 1 + \varepsilon$ and agent 2 has endowment $e_t = 1 - \varepsilon$, and whenever $z_t = L$, agent 1 has endowment $e_t = 1 - \varepsilon$ and agent 2 has endowment $e_t = 1 + \varepsilon$.
- $\pi(z_{t+1} = H | z_{t-1} = H) \geq \pi(z_{t+1} = L | z_{t-1} = H)$ and $\pi(z_{t+1} = H | z_{t-1} = L) \geq \pi(z_{t+1} = L | z_{t-1} = H)$. Notice that in the case these inequalities are valid with equality, we are in the iid environment.

We assume agents' utility is the same as before:

$$U(c)(z^t) \equiv \sum_{s=t}^{\infty} \sum_{z^s \in Z^s} \beta^t u(c_s(z^s))\pi(z^s|z^t),$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, strictly concave and $C^1$, and where $\beta \in (0,1)$ is the time invariant discount factor. We also assume the Inada condition $\lim_{c\to 0} u'(c) = \infty$ holds.

## 3.3 Kehoe and Levine Stationary Equilibrium

In this section, we characterize the stationary symmetric[1] equilibrium allocations for this economy. Using the same notation as Dirk Krueger and Fabrizio Perri (2005), we define $U(1+\varepsilon)$ and $U(1-\varepsilon)$ as the continuation utility from consuming the autarkic allocation, when the agent's current income is $1+\varepsilon$ and $1-\varepsilon$, respectively:

$$U(1+\varepsilon) = u(1+\varepsilon) + \sum_{t=1} \sum_{z^t \in Z^t} \beta^t \left(\pi(z^t|z_0 = H)\mathbb{1}_{(z_t=H)} u(1+\varepsilon) + \pi(z^t|z_0 = H)\mathbb{1}_{(z_t=L)} u(1-\varepsilon)\right)$$

$$U(1+\varepsilon) = u(1+\varepsilon) + u(1+\varepsilon)\underbrace{\sum_{t=1} \sum_{z^t \in Z^t} \beta^t \pi(z^t|z_0 = H)\mathbb{1}_{(z_t=H)}}_{<\infty} + u(1-\varepsilon)\underbrace{\sum_{t=1} \sum_{z^t \in Z^t} \beta^t \pi(z^t|z_0 = H)\mathbb{1}_{(z_t=L)}}_{<\infty}$$

and

$$U(1-\varepsilon) = u(1-\varepsilon) + u(1+\varepsilon)\underbrace{\sum_{t=1} \sum_{z^t \in Z^t} \beta^t \pi(z^t|z_0 = L)\mathbb{1}_{(z_t=H)}}_{<\infty} + u(1-\varepsilon)\underbrace{\sum_{t=1} \sum_{z^t \in Z^t} \beta^t \pi(z^t|z_0 = L)\mathbb{1}_{(z_t=L)}}_{<\infty},$$

Let $U^{FB}(1)$ be the first best allocation at which both agents consume 1 unit of the good at each period. Then we have the following results:

**Lemma 3.3.1** *$U(1+\varepsilon)$ is strictly increasing in $\varepsilon$ at $\varepsilon = 0$, is strictly decreasing in $\varepsilon$ as $\varepsilon \to 1$, and is strictly concave in $\varepsilon$, with a unique maximum*

$$\varepsilon_1 = \arg\max_{\varepsilon} U(1+\varepsilon) \in (0,1).$$

*Moreover, $U(1+\varepsilon_1) > U^{FB}(1)$ and there exists $0 < \bar{\varepsilon} \leq 1$ such that $U(1+\bar{\varepsilon}) \geq U^{FB}(r)$. Consequently, $U(1+\varepsilon) > U^{FB}(1)$ for $\varepsilon \in (0,\bar{\varepsilon})$, and thus, for these $\varepsilon$, complete risk sharing is worse than autarky for the currently rich agent.*

**Proof:** See the Appendix. ■

[1]Without loss of generality, we can assume that the first best allocation as well as the equilibrium allocation are symmetric, as long as we make the proper redistribution of the agents initial assets.

**Proposition 3.3.2** *The constrained efficient symmetric consumption distribution is completely characterized by a number* $0 \leq \varepsilon_c(\varepsilon) \leq \varepsilon$. *Agents with labor income* $1+\varepsilon$ *consume* $1+\varepsilon_c(\varepsilon)$, *and agents with income* $1-\varepsilon$ *consume* $1-\varepsilon_c(\varepsilon)$. $\varepsilon_c(\varepsilon)$ *is the smaller nonnegative solution to the following equation*

$$U(1+\varepsilon_c(\varepsilon)) = \max\{U^{FB}(1), U(1+\varepsilon)\},$$

*and* $U(1+\varepsilon)$ *is the lifetime utility of the currently rich agent from the consumption allocation characterized by* $\varepsilon_c(\varepsilon)$.

**Proof:** See the Appendix. ■

**Proposition 3.3.3** *For* $\varepsilon \in [\overline{\varepsilon}, 1)$ *perfect insurance is feasible and a marginal increase in* $\varepsilon$ *has no effect on consumption inequality. If* $\varepsilon \in [\varepsilon_1, \overline{\varepsilon})$, *then an increase in* $\varepsilon$ *decreases consumption inequality. For* $\varepsilon \in [0, \varepsilon_1)$, *autarky is the equilibrium allocation, so that an increase in* $\varepsilon$ *increases consumption inequality by the same amount.*

**Proof:** See the Appendix. ■

Proposition 3.3.3 shows that an increase in income inequality may not alter or even reduce consumption dispersion (for $\varepsilon \in [\overline{\varepsilon}, 1) \cup [\varepsilon_1, \overline{\varepsilon})$).

# Appendix B

**Proof of lemma 3.3.1:**

i) Taking the derivative of $U(1+\varepsilon)$ with respect to $\varepsilon$ at $\varepsilon = 0$, we have that

$$\begin{aligned}\left.\frac{\partial U(1+\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} &= u'(1) + u'(1)\sum_{t=1}\sum_{z^t\in Z^t}\beta^t\pi(z^t|z_0=H)\mathbb{1}_{(z_t=H)} - u'(1)\sum_{t=1}\sum_{z^t\in Z^t}\beta^t\pi(z^t|z_0=H)\mathbb{1}_{(z_t=L)} \\ &= \underbrace{u'(1)}_{>0} + \sum_{t=1}\sum_{z^{t-1}\in Z^{t-1}}\underbrace{\beta^t\pi(z^{t-1}|z_0=H)}_{>0}[\pi(z_t=H|z_{t-1})-\pi(z_t=L|z_{t-1})]\underbrace{u'(1)}_{>0}. \end{aligned} \tag{B.1}$$

Using our assumptions about the probabilities $\pi(z_t|z_{t-1})$, it is easy to show that the terms in brackets in equation B.1 are always positive. Indeed, if $z_{t-1} = H$, then, by the assumption that $\pi(z_{t+1} = H|z_{t-1} = H) \geq \pi(z_{t+1} = L|z_{t-1} = H)$, we have that

$$[\pi(z_{t+1} = H|z_{t-1} = H) - \pi(z_{t+1} = L|z_{t-1} = H)] \geq 0,$$

If, on the other hand, $z_{t-1} = L$, then, by the assumption that $\pi(z_{t+1} = H|z_{t-1} = L) \geq \pi(z_{t+1} = L|z_{t-1} = H)$, we have that

$$[\pi(z_{t+1} = H|z_{t-1} = L) - \pi(z_{t+1} = L|z_{t-1} = H)] \geq 0.$$

Therefore, $\left.\frac{\partial U(1+\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} > 0$.

ii) Deriving $U(1+\varepsilon)$ as $\varepsilon \to 0$, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 1}\frac{\partial U(1+\varepsilon)}{\partial \varepsilon} &= u'(2) + u'(2)\sum_{t=1}\sum_{z^t\in Z^t}\beta^t\pi(z^t|z_0=H)\mathbb{1}_{(z_t=H)} - \lim_{\varepsilon\to 1}u'(1-\varepsilon)\sum_{t=1}\sum_{z^t\in Z^t}\beta^t\pi(z^t|z_0=H)\mathbb{1}_{(z_t=L)} \\ &= u'(2) + u'(2)\sum_{t=1}\sum_{z^t\in Z^t}\beta^t\pi(z^t|z_0=H)\mathbb{1}_{(z_t=H)} - \underbrace{\lim_{c\to 0}u'(c)}_{\infty}\sum_{t=1}\sum_{z^t\in Z^t}\beta^t\pi(z^t|z_0=H)\mathbb{1}_{(z_t=L)} \\ &= -\infty. \end{aligned}$$

iii) Taking the second derivative of $U(1+\varepsilon)$ with respect to $\varepsilon$, one easily obtains

$$\frac{\partial^2 U(1+\varepsilon)}{\partial \varepsilon^2} = \underbrace{u''(1+\varepsilon)}_{<0} + \underbrace{u''(1+\varepsilon)}_{<0} \left( \sum_{t=1} \sum_{z^t \in Z^t} \beta^t \pi(z^t | z_0 = H) \mathbb{1}_{(z_t = H)} \right)$$
$$+ \underbrace{u''(1-\varepsilon)}_{<0} \left( \sum_{t=1} \sum_{z^t \in Z^t} \beta^t \pi(z^t | z_0 = H) \mathbb{1}_{(z_t = L)} \right) < 0.$$

iv) From i, ii and iii, it follows immediately that there is a unique maximum

$$\varepsilon_1 = \arg\max_{\varepsilon} U(1+\varepsilon) \in (0,1).$$

v) $U(1+\varepsilon_1) > U^{FB}(1)$ is obvious, since $U(1+0) > U^{FB}(1)$ and $\left.\frac{\partial U(1+\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} > 0.$

vi) If $U(2) \geq U^{FB}(1)$, then, setting $\overline{\varepsilon} = 1$, we have that $U(1+\overline{\varepsilon}) = U(2) \geq U^{FB}(1)$. If $U(2) < U^{FB}(1)$, then, since $U(1+\varepsilon) > U^{FB}(1) \quad \forall \varepsilon \in (0, \varepsilon_1]$ and since $U(1+\varepsilon)$ is continuous in $\varepsilon$, we have by the intermediate value theorem, that there is an $\overline{\varepsilon} \in (\varepsilon_1, 1)$ such that $U(1+\overline{\varepsilon}) = U^{FB}(1)$. The concavity of $U(\cdot)$ ensures that there is at most one $\overline{\varepsilon} > 0$ such that $U(1+\overline{\varepsilon}) = U^{FB}(1)$.

■

**Proof of proposition 3.3.2:**

Since both agents want to smooth out consumption, they both wish to choose the smallest $\varepsilon_c \in [0,1]$ such that the participation constraint is satisfied. Therefore, if $U(1+\varepsilon) < U^{FB}(1)$, the solution is simply $\varepsilon_c = 0$.

If $U(1+\varepsilon) > U^{FB}(1)$, then the agent will choose the smallest $\varepsilon_c \in (0,1]$ such that $U(1+\varepsilon_c) \leq U(1+\varepsilon)$. Suppose by contradiction that the optimal allocation $\varepsilon_c$ is such that $U(1+\varepsilon_c) < U(1+\varepsilon)$. Then, by the continuity of $U(\cdot)$, there is an $\varepsilon'_c < \varepsilon_c$ such that $U(1+\varepsilon'_c) < U(1+\varepsilon)$, a contradiction with $\varepsilon_c$ being optimal. ■

**Proof of proposition 3.3.3:**

i) By proposition 3.3.2, we have that, for $\varepsilon \in [\overline{\varepsilon}, 1)$ there is perfect risk sharing, which implies that a marginal increase in $\varepsilon$ in this interval does not affect consumption inequality.

ii) Since $U(1+\varepsilon) > U^{FB}(1)$ and $U(1+\varepsilon)$ is decreasing in $\varepsilon$ for $\varepsilon \in [\varepsilon_1, \overline{\varepsilon})$, an increase in $\varepsilon$ reduces $U(1+\varepsilon)$, which implies that $\varepsilon$, implicitly defined by

$$U(1+\varepsilon_c(\varepsilon)) = U(1+\varepsilon),$$

must decrease.

iii) Since $U(1+\varepsilon) > U^{FB}(1)$ and $U(1+\varepsilon)$ is increasing in $\varepsilon$ for $\varepsilon \in [0, \varepsilon_1))$, the equilibrium allocation must be the autarkic one, $\varepsilon_c = \varepsilon$, so that an increase in $\varepsilon$ implies an equal increase in $\varepsilon_c$.

■

# Chapter 4

# Credit limit with small costs of default

## 4.1 Introduction

Self enforcing debt constraint are defined in the literature as the debt limit generated by agents' participation constraints (Bloise and Reichlin, 2009). The participation constraints, in turn, are determined by the kind of punishment that is applied to bankrupt agents in the economy. The stronger the punishment, the greater is the capacity of enforcement in the economy and, therefore, the greater is the debt limit for each agent.

In Kehoe and Levine (1993) and Alvarez and Jermman (2000) environment, the punishment for bankruptcy was the permanent exclusion from the asset market. Motivated by the fact that this may seem to be a too harsh and unrealistic punishment for default, some authors have adapted Alvarez and Jermman (2000) model by assuming smaller default costs, which implies tighter solvency constraints.

By doing so, these authors sometimes allow the punishment for default to depend on asset prices. Since prices are endogenously determined by the equilibrium conditions, the participation constraints of these models may be endogenous. Endogenous participation constraints, on the other hand, may prevent us from using the solution to a planner's problem in order to find the equilibrium consumption allocations of the decentralized problem, since the planner's problem will also depend on (endogenous) prices.

We are going to present two of these models with smaller costs of default in this chapter. The first, which is being developed by Azariadis and Kaas, consists on a dynamic general equilibrium model where the exclusion from the asset market lasts only a finite number of periods. They show that, by allowing defaulting agents to reenter asset markets after a fixed exclusion period, we have that: (i) the first best allocation may not be implemented if individuals are arbitrarily patient, (ii) there is dynamic complementarity in prices which permits multiple steady states to coexist, and in turn contributes to economic volatility, (iii) the autarkic equilibrium can be robust to the introduction of additional small costs of default.

The second model, developed by Hellwig and Lorenzoni (2009), consists of a dynamic general equilibrium model where punishment from default consists on the permanent exclusion from borrowing. That is, once an agent commits default, he can not borrow ever again, but he can lend money at market prices. Their main conclusion is that, by allowing defaulting agents to save money, in an equilibrium where there are some agents borrowing money in some states, the implied interest rate must be low.

## 4.2 Sequential equilibrium with temporary exclusion (Azariadis and Kaas (2008))

This section presents the first version of the Azariadis and Kaas (2008) working paper.

### 4.2.1 Deterministic case with two agents

Consider the same environment described in section 2.2, except that now:

- $\mathbf{I} = \{0, 1\}$, i.e., there are only two agents.
- Endowments are non stochastic and fluctuate between a high level normalized to unity, and a low level $\theta < 1$ according to:

$$e_{0,t} = 1 \quad e_{1,t} = \theta \quad \text{for all even } t$$
$$e_{0,t} = \theta \quad e_{1,t} = 1 \quad \text{for all odd } t$$

**Definition 4.2.1** *(Azariadis ans Kaas (2008), page 5) An equilibrium with limited commitment with exclusion length L is a list of consumption plans and asset holdings* $(c_{i,t}, a_{i,t})_{t\geq 0}$ *for* $i = 0, 1$, *credit limits* $(B_{i,t})_{t\geq 0}$ *for* $i = 0, 1$ *and security prices* $(q_t)_{t\geq 0}$ *such that*

*(i) given* $(q_t)_{t\geq 0}$, *for each* $i$, $(c_{i,t}, a_{i,t})_{t\geq 0}$ *solves*

$$J_{i,t}(a_{i,t}) = \max_{c_{i,t}, \{a_{i,t+1}\}} \{u(c_{i,t}) + \beta J_{i,t+1}(a_{i,t+1})\}, \tag{4.1}$$

*subject to*

$$e_{i,t} + a_{i,t} = a_{i,t+1} q_t + c_{i,t}, \tag{4.2}$$
$$a_{i,t+1} \geq B_{i,t+1} \tag{4.3}$$

*(ii) Markets clear, i.e., for all* $t \geq 0$,

$$c_{0,t} + c_{1,t} = 1 + \theta \quad \textit{and} \quad a_{0,t} + a_{1,t} = 0.$$

*(iii) Short-sale constraints prevent default: for any default date $t \geq 1$ and $i = 0, 1$, the solvency payoff from $t$ forward is no smaller than the default payoff, that is,*

$$U_t(c_i) \equiv \sum_{\tau \geq t} \beta^{\tau - t} u(c_{i,\tau}) \geq \overline{U}_{t,i}, \tag{4.4}$$

*where the default payoff $\overline{U}_{i,t}$ maximizes $\sum_{\tau \geq t} \beta^{\tau-t} u(c_{i,\tau})$ subject to*

$$\begin{aligned} e_{i,t} + a_{i,t} \geq a_{i,t+1} q_t + c_{i,t}, &\quad \tau \geq t \\ a_{i,t} = 0, &\quad t \leq \tau \leq t + L - 1 \\ a_{i,t} \geq B_{i,t}, &\quad \tau \geq t + L \end{aligned}$$

*(iv) Short sale constraints are not too tight, i.e., whenever $a_{i,t-1} \geq B_{i,t-1}$ binds in problem (i), the participation constraint 4.4 is satisfied with equality in period $t$. Moreover, for defaulting agents, the participation constraint $\sum_{k \geq \tau} \beta^{k-\tau} u(\bar{c}_{i,k}) \geq \overline{U}_{i,\tau}$ binds whenever the borrowing constraint $\bar{a}_{i,\tau-1} \geq B_{i,\tau-1}$ in the utility maximization problem of (iii).*

Let:

- $x_t$ denote the consumption of a high-income agent in period $t$.
- $b_t$ denote borrowing of a low-income agent in $t$.
- $B_t$ denote the credit limit of a low-income agent in period $t$.

Since credit limits are never positive, and since high-income agents supply credit, we have that high-income agents are unconstrained, which implies the Euler equation can be written as[1]:

$$q_t = \beta \frac{u'(1 + \theta - x_{t+1})}{u'(x_t)}. \tag{4.5}$$

$$\geq \beta \frac{u'(x_{t+1})}{u'(1 + \theta - x_t)}, \quad \text{with strict inequality iff } b_t = B_t. \tag{4.6}$$

and the budget constraint of a high-income agent can be written as

$$1 - b_{t-1} \geq x_t + q_t b_t \tag{4.7}$$

**Proposition 4.2.1** *(Azariadis ans Kaas (2008), page 7) The credit limit for the low income agent binds iff $x_t + x_{t+1} > 1 + \theta$.*

[1] The Euler equation can be derived here using the same steps we used to derive the Euler equation for a conventional A-J equilibrium.

**Proof:** See the Appendix. ■

Now, let us derive some conditions under which the first best allocation is implementable. We are going to analyze the implementation of a symmetric first best allocation, where consumption equals $c_{i,t} = \frac{1+\theta}{2}$ for all individuals at all dates[2]. Therefore, our equilibrium candidates are $c_{i,t} = \frac{1+\theta}{2}$ and $q_t = \beta \ \forall t$.

**Proposition 4.2.2** *(Azariadis ans Kaas (2008), page 7) Suppose that exclusion lasts an odd number of periods, $L = 2m+1$ with $m \geq 0$ and suppose an agent defaults in some high income period (say t=0). Then, when he reenters the asset market in period $2m+1$ his credit limit binds, and from period $2m+2$ on he never borrows again and achieves flat consumption at*

$$\hat{c}(B) = \frac{1+\beta\theta}{1+\beta} + B(1-\beta),$$

*which has the same present value as the net income vector $(1-z, \theta, 1, \theta, \cdots)$.*

From proposition 4.2.2, we have that the largest borrowing limit $B$ which ensures no default occurs in $t = 2m+2$, is the one that solves

$$\frac{u[\hat{c}(B)]}{1-\beta} = \frac{1-\beta^{2m}}{1-\beta^2}\left[u(1)+\beta(\theta)\right] + \beta^{2m}\left[u(1)+\beta u(\theta-\beta B)\right] + \beta^{2m+2}\frac{u[\hat{c}(B)]}{1-\beta} \quad (4.8)$$

Furthermore, the fist best allocation $c_{i,t} = \frac{1+\theta}{2}$ is implementable iff

$$\begin{aligned}
&\frac{1+\theta}{2} \geq \hat{c}(B) = \frac{1+\beta\theta}{1+\beta} + B(1-\beta)\\
&\iff B \leq \frac{1-\theta}{2(1+\beta)}\\
&\iff u\left(\frac{1+\theta}{2}\right) \geq u(\hat{c}(B))\\
&\overset{(4.8)}{\iff} (1+\beta)u\left(\frac{1+\theta}{2}\right) \geq \frac{1-\beta^{2m}}{1-\beta^{2m+2}}\left[u(1)+\beta(\theta)\right] + \frac{\beta^{2m}-\beta^{2m+2}}{1-\beta^{2m+2}}\left[u(1)+\beta u\left(\frac{2\theta+\beta+\theta\beta}{2(1+\beta)}\right)\right]
\end{aligned}$$

Now let us analyze a steady state equilibrium with binding debt limits for low income-agents[3]. Let $x$ be a stationary consumption of high income agent. Then security price must be given by $\beta u'(1+\theta-x)/u'(x)$. Moreover, if the participation constraint of the low-income agent is always binding, then

$$\begin{aligned}
0 = J(x) \equiv & u(x) + \beta u(1+\theta-x) - \frac{1-\beta^{2m}}{1-\beta^{2m+2}}\left[u(1)+\beta u(\theta)\right]\\
& - \frac{\beta^{2m}-\beta^{2m+2}}{1-\beta^{2m+2}}\left\{u(1)+\beta u\left[\theta+q(x)b(x)\right]\right\}
\end{aligned}$$

[2]Clearly, the individual with lowest consumption will have more incentives to default. Therefore, the implementability of a symmetric first best allocation is a necessary condition for the implementability of any arbitrary asymmetric first best allocation, where one of the agents gets to consume less than the other agent at every period.

[3]Notice that if a stationary equilibrium is not binding, then it's corresponding equilibrium allocation is necessarily the first best allocation.

where

$$q(x)b(x) = \frac{\beta u'(1+\theta-x)(1-x)}{u'(x)+\beta u'(y-x)} \tag{4.9}$$

Notice that, whenever $J(\frac{1+\theta}{2}) > 0$, the first best allocation is implementable.

The figures bellow illustrate, for some choice of parameters, when the first best equilibrium is attainable. Notice that, in both graphics, autarky is always an equilibrium. However, in the infinite exclusion case, autarky is only robust when it is the only stationary equilibrium, whereas autarky is always a robust equilibrium for the case exclusion lasts only $L = 1$ periods.

Finally, we can see that, when $L = \infty$, there are at most two equilibria, whereas $L = 1$ sometimes allow three (robust) equilibria. According to Azariadis and Kaas (2008), this phenomena can be explained by the existence of price complementarity, that is only present when exclusion lasts a finite number of periods. Indeed, defining $R_t \equiv 1/q_t$, we have that binding debt limits imply that

$$u(x_t) + \beta u(1+\theta-x_{t+1}) = u(1) + \beta u(1+\theta-x_{t+1}-R_t b_t). \tag{4.10}$$

Notice, from equation 4.10, that the penalty from default consists on not being able to save to the next period. It can be easily seen, from equation 4.10, that the penalty from default increases, as $R_t$ increases. Therefore, an increase in future interest rates $R_t$ increases the penalty for default today, which loosens individuals' solvency constraints, allowing them to borrow more today. As people borrow more today, interest rates today must increase in order to accommodate an increase in the demand for assets today, generating complementarity in interest rates (or equivalently, in prices).

Notice that this price complementarity mechanism is not present in the conventional model with infinite exclusion from the asset market, since binding solvency constraints when $L = \infty$ implies that

$$u(x_t) + \beta u(1+\theta-x_{t+1}) = u(1) + \beta u(\theta),$$

which doesn't depend on prices.

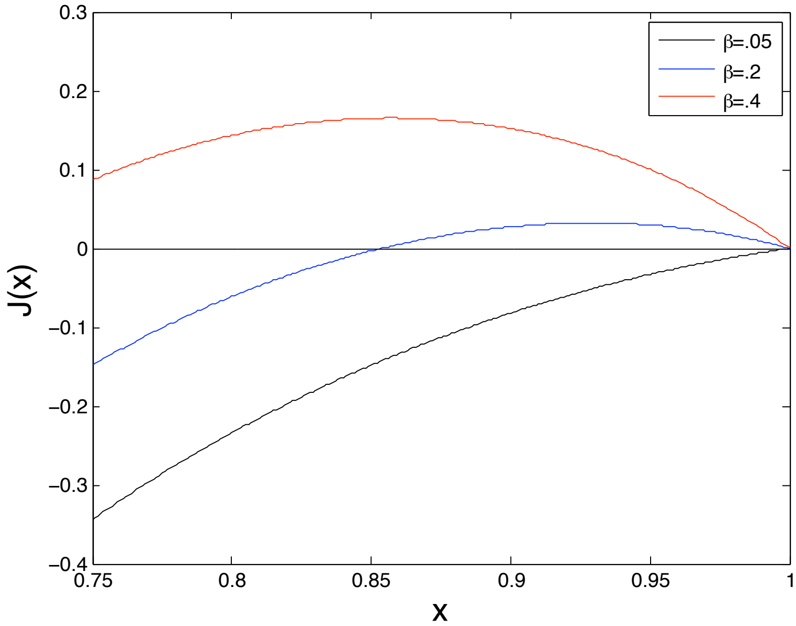


Figure 4.1: The curve $J(x)$ defining stationary equilibrium when $L = \infty$, $u(c) = c^{1-\gamma}$, $\gamma = 10/3$ and $\theta = .5$ and for three different discount factors.

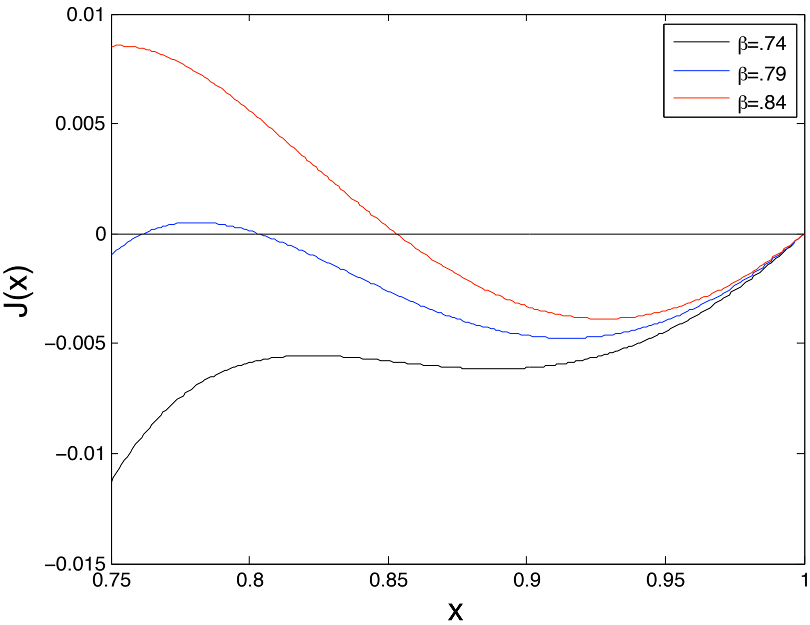


Figure 4.2: The curve $J(x)$ defining stationary equilibrium when $L = 1$, $u(c) = c^{1-\gamma}$, $\gamma = 10/3$ and $\theta = .5$ and for three different discount factors.

## 4.3 Sequential equilibrium with finite exclusion duration (Azariadis and Kaas (2012))

On this latest version of Azariadis and Kaas working paper, the authors develop an environment similar to the one described in the last section, except that now they assume that the defaulting agent can regain access to the asset market with probability $\mu$, for each subsequent period.

This formulation has the advantage of letting us specify the outside option of the consumer problem in a recursive way. Moreover, as we can notice from the first version of Azariadis and Kaas' working paper, the authors had only found interesting results, such as the existence of multiple stationary equilibria, for one period exclusion. In this new formulation, however, they find multiple equilibria for high values of $\mu$ that are lower than one (i.e., when the expected exclusion duration for defaulters is small, but higher than one).

But their results are qualitative the same as the ones derived in the first version of their working paper: (i) multiple (robust) stationary consumption equilibria may exist when the expected exclusion duration is finite (i.e., $\mu > 0$), whereas there is at most one stationary trading equilibrium when the duration is infinite (i.e., $\mu = 0$), (ii) when the expected exclusion duration is finite (i.e., $\mu > 0$), the symmetric first best allocation may not be implementable for values of $\beta$ arbitrarily close to one, whereas, when the expected exclusion duration is infinite (i.e., $\mu = 0$), for any given parameters, there exists a sufficiently high value of $\beta$ for which the symmetric first best allocation is implementable, (iii) the autarkic equilibrium can be robust to the introduction of additional small costs of default.

### 4.3.1 Environment

Assume that there is a continuum of infinitely lived consumers $i \in [0, 1]$. There is one nondurable good in the economy and at each period, each individual $i \in [0, 1]$ receives a shock $z_{it} \in \{H, L\}$ that determines his endowment of the nondurable good $e_i(z_{it})$. We assume that the shocks follow a Markov process with transition probabilities:

$$\pi(H|H) = \pi_H \quad \pi(L|H) = 1 - \pi_H$$
$$\pi(L|L) = \pi_L \quad \pi(H|L) = 1 - \pi_L.$$

When $z_{it} = H$ individual $i$ has high income and when and when $z_{it} = L$ individual $i$ has low income. Specifically, we assume that

$$e_i(H) = \lambda > 1$$
$$e_i(L) = 1 - (\lambda - 1)\frac{1 - \pi_L}{1 - \pi_H} < 1,$$

so that average stationary income is 1.

Using conventional notation, we write $z_i^t$ as the history of shocks for agent $i$ and we assume that agents have a conventional separable expected utility function

$$\sum_{t=0} \sum_{z_i^t} \beta^t \pi(z_i^t) u(c_i(z_i^t)),$$

where $\beta \in (0,1)$, and where $c_i(z_i^t)$ denotes the consumption of individual $i$ at date $t$, history $z_i^t$.

We assume that at each period, agents transact a complete set of one period contingent assets. Let $a_i(z_i^t, z)$ be the quantity of assets purchased by individual $i$ at date $t$ history $z_i^t$ that promises to pay one unit of the nondurable if the next period is $z_i$, and let $q(z_i^t, z_i)$ be the price of this asset.

We assume that a agents who decides to default can regain access to the asset market in the next periods with probability $\mu$. Therefore, the expected duration of market exclusion for a defaulting agent is given by $1/\mu$.[4]

### 4.3.2 Definition of Equilibrium

For notational ease, we omit from now on the $i$ script of each agent's shock. Then the concept of equilibrium for this economy is defined as:

**Definition 4.3.1** *(Azariadis ans Kaas (2012), page 6) An* ***equilibrium with limited commitment and duration*** $1/\mu$ *is a list of consumption plans and assets* $(c_{i,t}(z^t), a_{i,t}(z^t))_{(z^t)}$*, initial conditions* $(a_{i,0}, z_{i,0})$*, credit limits* $(b_{i,t}(z^t))_{(z^t)}$ *and prices* $(q_t(z^t))_{(z^t)}$ *such that:*

*(i) given* $(q_t(z^t))_{(z^t)}$, $(c_{i,t}(z^t), a_{i,t}(z^t))_{(z^t)}$ *solves*

$$
\begin{aligned}
& U_t(a, z^t) = \max_{c, a(H), a(L)} \left\{ u(c) + \beta \sum_{z'=H,L} \pi(z'|z_t) U_t(a(z'), (z^t, z')) \right\}, \\
s.t. \quad & e(z_t) + a = \sum_{z'=H,L} a(z') q_t(z^t, z') + c, \\
& a(z') \geq -\bar{b}(z^t, z')
\end{aligned}
$$

*(ii) There is market clearing in the consumption market and in the asset market:*

$$
\int_0^1 \sum_{z^t} \pi(z^t|z_{i,0}) c_{i,t}(z^t)\, di = 1 \quad e \quad \int_0^1 \sum_{z^t} \pi(z^t|z_{i,0}) a_{i,t}(z^t)\, di = 0 \quad \forall t
$$

*(iii) The participation constraints are satisfied:*

$$
U_t(a_{i,t}(z^t), z^t) \geq \overline{U_t}(z^t),
$$

*where* $\overline{U_t}(z^t)$ *is recursively defined by*

$$
\overline{U}_t(z^t) = u(e(z_t)) + \beta\mu \sum_{z=H,L} \pi(z|z_t) U_t(0, (z^t, z)) + \beta(1-\mu) \sum_{z=H,L} \pi(z|z_t) \overline{U}_t(z^t, z).
$$

[4]Elementary algebra can be used to show that the expected exclusion duration is given by $1\mu + 2\mu \cdot (1-\mu) + 3\mu \cdot (1-\mu)^2 + \cdots = 1/\mu$.

*(iv) The **solvency constraints are not too tight:** $a_{i,t}(z^t) \geq -b_{i,t}(z^t)$ is binding in problem (I) iff $U_t(a_{i,t}(z^t), z^t) \geq \overline{U_t}(z^t)$ is binding.*

Now notice that, since the first part of this definition of equilibrium is exactly the same as the one developed by Alvarez and Jermann (2000), their Euler equation and transversality conditions equally apply here. Therefore, the sufficient conditions for optimization for this problem are given by

$$q_t(z^t, z') \geq \frac{\beta\pi(z'|z)u'(c_{i,t}(z^t, z'))}{u'(c_{i,t}(z^t))}, \qquad \text{(Euler Equation)}$$

with equality if $a_{i,t}(z^t, z') > -b_{i,t}(z^t, z')$, and

$$\lim_{t\to\infty} \sum_{z^t \in \{H,L\}^t} \beta^t u'(c_{i,t}(z^t))[a_{i,t}(z^t) + \bar{b}_{i,t}(z^t)]\pi(z^t|z_{i,0}) = 0. \qquad \text{(Transv. Cond.)}$$

We make an observation here that in Alvarez and Jermann (2000), we have that credit limits are always positive, that is, $\bar{b}_{i,t}(z^t) \geq 0$. This result follows from the fact that, if the individual defaults, then the allocation is feasible, and from the fact that $U_t(\cdot, z^t)$ is increasing. However, Azariadis and Kaas (2012) don't present a formal proof that $U(0, z^t) \geq \overline{U}_t(z^t)$, which would guarantee that credit limits would always be positive in their formulation with endogenous outside option.

### 4.3.3 Stationary Equilibria

In this section we derive sufficient conditions for the existence of a stationary equilibrium. We say an equilibrium is stationary if consumption allocations, asset holdings and prices only depend on the individual current shock, and if the distribution of agents' types is constant across states.

To find the stationary proportion of high income individuals, $\phi_H$, and low income individuals $\phi_L$, one only needs to solve

$$\begin{pmatrix} \pi_H & 1-\pi_L \\ 1-\pi_H & \pi_L \end{pmatrix} \begin{pmatrix} \phi_H \\ \phi_L \end{pmatrix} = \begin{pmatrix} \phi_H \\ \phi_L \end{pmatrix}$$

to obtain

$$\varphi_H \equiv \frac{1-\pi_L}{2-\pi_H-\pi_L}$$

and

$$\phi_L \equiv 1 - \phi_H.$$

Let:

- $x \in [1, \lambda]$ be the stationary consumption of a high income agent;
- $c_L(x) \equiv 1-(x-1)\frac{1-\pi_L}{1-\pi_H} \in [0,1]$ be the stationary consumption of a low income agent;
- $q_{z,z'}$ the price an individual in state $z$ pays for an asset that promises one unit of consumption in state $z'$ in the next period;

- $a_{z,z'}$ be the quantity of assets that promises one unit of consumption in the next period contingent on the realization of $z'$, that an individual at $z$ holds.

Then, using the Euler Equations, budget constraints and market clearing conditions, we can prove the following result:

**Lemma 4.3.1** *(Azariadis ans Kaas (2012), page 7) Let $x \in [1, \lambda]$ be the consumption of a high income agent, $c_L(x) \equiv 1 - (x-1)\frac{1-\pi_L}{1-\pi_H}$ be the consumption of a low income agent in a Markov stationary equilibrium. Then equilibrium prices are given by*

$$q_{LL} = \beta\pi_L, \quad q_{HL}(x) = \frac{\beta(1-\pi_H)u'(c_L(x))}{u'(x)}$$
$$q_{HH} = \beta\pi_H, \quad q_{LH}(x) = \frac{\beta(1-\pi_L)}{1-\pi_H}\left[\pi_L - \pi_H + (1-\pi_L)\frac{u'(c_L(x))}{u'(x)}\right],$$

*asset holdings equilibrium are given by*

$$a_{LH} = a_{HH} = -b(x), \quad a_{HL} = a_{LL} = \frac{1-\pi_L}{1-\pi_H}b(x),$$

*where*

$$b(x) \equiv \frac{(\lambda - x)}{1 - \beta\pi_H + \beta(1-\pi_L)\frac{u'(c_L(x))}{u'(x)}}.$$

Notice that $b(x)$ is decreasing in $x$. If we assume that the credit limits are binding for low income agents, this reflects the fact that the greater is the consumption smoothing (the closer $x$ is from 1), the tighter are the solvency constraints (the lower are the credit limits).

Also notice that, if $x > c_L(x)$, then we must have binding solvency constraints for low income agents buying assets for high income periods. This follows from the fact that

$$q_{LH} > \beta(1-\pi_L)\frac{u'(x)}{u'(c_L(x))},$$

as one can easily verify.

It is important emphasizing that the conditions stated in the preceding lemma are only necessary, but not sufficient to guarantee the existence of equilibrium. For the allocations and prices defined in the preceding lemma to be an equilibrium we have to find not too tight solvency constraints and verify that they are satisfied.

### 4.3.4 Implementing the first best allocation

As in the first version of their working paper, when studying the implementation of first best allocations, Azariadis and Kaas (2012) focus on the implementation of the symmetric stationary equilibrium, where consumption of every agent is constant over time and equal to the average income of 1. They do this because: (1) If an asymmetric first best equilibrium is implementable, then the symmetric first best equilibrium is implementable as long as we make an appropriate redistribution of initial wealth; (2) A necessary condition for an asymmetric first best equilibrium to be implementable is that the symmetric first best equilibrium is implementable for an appropriate redistribution of initial assets[5].

In order for the symmetric first best allocation $x = c_L(x) = 1$ to be an equilibrium, by lemma 4.3.1, we must have that:

- $q_{zz} = \beta\pi_z$, $q_{zz'} = \beta(1-\pi_z)$, for all $z \neq z' \in \{H, L\}$.
- The utility from staying solvent is given by

$$U^* = u(1)/(1-\beta).$$

- The utility from defaulting in the good state is

$$\overline{U}_H = u(\lambda) + \beta(1-\mu)\left[\pi_H\overline{U}_H + (1-\pi_H)\overline{U}_L\right] + \beta\mu\left[\pi_H U_H^0 + (1-\pi_H)U_L^0\right]$$

- The utility from giving default in the bad state is

$$\overline{U}_L = u(e(\lambda)) + \beta(1-\mu)\left[\pi_L\overline{U}_L + (1-\pi_L)\overline{U}_H\right] + \beta\mu\left[\pi_L U_L^0 + (1-\pi_L)U_H^0\right]$$

If these conditions are satisfied, the following lemmas can be proved (see Azariadis ans Kaas (2012), pages 8-9):

**Lemma 4.3.2** *An individual who defaults and regains access to the asset market in a good state, chooses a constant level of consumption from there one and equal to*

$$\begin{aligned} c_H^0 &= \frac{\lambda(1-\beta)+\beta(2-\beta_H-\beta_L)}{1+\beta(1-\pi_L-\pi_H)} > 1 \\ \Rightarrow \quad U_H^0 &= \frac{1}{1-\beta}u\left(\frac{\lambda(1-\beta)+\beta(2-\beta_H-\pi_L)}{1+\beta(1-\pi_L-\pi_H)}\right). \end{aligned} \tag{III}$$

**Lemma 4.3.3** *$\overline{b}$ is defined such that an agent who enters in the good state with income $\lambda - \overline{b}$ is exactly indifferent between defaulting and staying solvent. If he stays solvent, he chooses to consume*

$$\overline{c} = c_H^0 - \overline{b}(1-\beta) \quad \Rightarrow \quad \frac{u\left(c_H^0 - \overline{b}(1-\beta)\right)}{1-\beta} = \overline{U}_H. \tag{IV}$$

*every period.*

**Lemma 4.3.4** *An agent who defaults and regains access to the asset market in the bad state chooses in the subsequent bad income states:*

$$\begin{aligned} c_L^0 &= e(L) + q_{LH}\overline{b} \\ a_{LH} &= -\overline{b} \quad a_{LL} = 0 \\ \Rightarrow \quad U_L^0 &= u(c_L^0) + \beta\pi_L U_L^0 + \beta(1-\pi_L)\overline{U}_H. \end{aligned} \tag{V}$$

[5]Indeed, if the stationary allocation is not symmetric, then it is more difficult to satisfy the credit constraints for the agents with lowest consumption.

**Lemma 4.3.5** *There is a unique solution $(\overline{U}_H, \overline{U}_L, U_H^0, U_L^0, \overline{b})$ for the system formed by equations (I), (II), (III), (IV) e (V).*

Agents do not have incentives to deviate from the first best allocation if and only if

$$u(1) \geq \overline{U}_H/(1-\beta) = u(\overline{c}).$$
$$\iff 1 \geq \overline{c} = c_H^0 - \overline{b}(1-\beta) \iff \overline{b} \geq b(1)$$

Using lemmas 4.3.2 to 4.3.5, Azariadis and Kaas (2012) prove the following theorem:

**Theorem 4.3.6** *(Azariadis ans Kaas (2012), page 9) An allocation is first best iff*

$$u(1) \geq \alpha_1 u(\lambda) + \alpha_2 u(e(L)) + \alpha_3 u\left(\frac{\lambda(1-\beta)+\beta(2-\pi_H-\pi_L)}{1+\beta(1-\pi_L-\pi_H)}\right) +$$
$$\alpha_4 u\left(e(L) + \frac{\beta(1-\pi_L)(\lambda-1)}{1+\beta(1-\pi_L-\pi_H)}\right),$$

*where $\alpha_i$ depends on the parameters $(\beta, \mu, \pi_H, \pi_L)$ and satisfy $\sum_{i=1}^4 \alpha_i = 1$.*

It can be shown that the right hand side of equation 4.3.6 is an expected utility of a lottery with expected payoff greater than 1. Therefore, we can not guarantee that this condition will be satisfied.

Notice also that as $\beta \to 1$, the inequality in 4.3.6 becomes innocuous: $u(1) \geq u(1)$. Therefore, in order to evaluate wether the first best allocation is implementable, Azariadis and Kaas (2012) compute, for different parameters, the derivative of the right hand side of the inequality in theorem 4.3.6 with respect to $\beta$ evaluated at $\beta = 1$, and verify whether the derivative is positive or negative. If the derivative is negative (positive, resp.), then, for $\beta$'s arbitrarily close to 1, the inequality in 4.3.6 is not satisfied (is satisfied, resp.).

If, for example,

$$u(c) = \frac{c^{(1-\sigma)}}{1-\sigma},$$

Azariadis and Kaas (2012) compute the following table, which displays the smallest risk aversion coefficient $\sigma$ for which the symmetric first best equilibrium is implementable when the duration is $1/\mu$ and $\beta$ is arbitrarily close to 1:

| $1/\mu$ | 1 | 2 | 3 | 4 | 5 | 10 | 20 | 30 |
|---|---|---|---|---|---|---|---|---|
| $\sigma$ | 19.4 | 13.2 | 10.0 | 8.0 | 6.7 | 3.6 | 1.9 | 1.3 |

As the table displays, the lower is the exclusion duration, the more difficult is to implement the first best allocation for values of $\beta$ arbitrarily close to 1.

This results contrasts with the case where the exclusion duration is infinite, i.e., with the Alvarez and Jermann (2000) equilibrium. Indeed, if $\mu = 0$, theorem 4.3.6 condition becomes

$$u(1) \geq \frac{1-\beta\pi_L}{1+\beta(1-\pi_H-\pi_L)} u(\lambda) + \frac{\beta(1-\pi_H)}{1+\beta(1-\pi_H-\pi_L)} u(e(L)),$$

which, for any given parameters, is satisfied for $\beta$ sufficiently close to 1.

### 4.3.5 Binding debt limits

In an equilibrium with binding solvency constraints we have that:

- $x > 1 > c_L(x)$.
- Utility from consuming the optimal allocation $x$ is given by

$$U_H^*(x) = u(x) + \beta\pi_H U_H^*(x) + \beta(1-\pi_H)U_L^*(x),$$
$$U_L^*(x) = u(c_L(x)) + \beta\pi_L U_L^*(x) + \beta(1-\pi_L)U_H^*(x).$$

- Utility from defaulting is given by

$$\overline{U}_H(x) = u(\lambda) + \beta(1-\mu)\left[\pi_H\overline{U}_H(x) + (1-\pi_H)\overline{U}_L(x)\right] + \beta\mu\left[\pi_H\tilde{U}_H(0,x) + (1-\pi_H)\tilde{U}_L(0,x)\right]$$
$$\overline{U}_L(x) = u(e(L)) + \beta(1-\mu)\left[\pi_L\overline{U}_L(x) + (1-\pi_L)\overline{U}_H(x)\right] + \beta\mu\left[\pi_L\tilde{U}_L(0,x) + (1-\pi_L)\tilde{U}_H(0,x)\right]$$

Where,

$$\tilde{U}_z(a,x) = \max_{\substack{a_{z,L}\geq -\bar{b}(z,L)\\ a_{z,H}\geq -b(x)}} \{u\left(e(z) + a - q_{zH}(x)a_{zH} - q_{zL}(x)a_{zL}\right) + \\ + \beta\left[\pi_z\tilde{U}_z(a_{zz},x) + (1-\pi_z)\tilde{U}_{z'}(a_{zz'},x)\right]\}.$$

An stationary Markov equilibrium allocation $x \in [1,\lambda]$ must satisfy

$$J(x) \equiv U_H^*(x) - \overline{U}_H(x) \geq 0, \qquad (*)$$

with equality if $x > 1$.

Azariadis and Kaas (2012) compute $J(\cdot)$ for values of $x$ close to $\lambda$ (autarky), and conclude that there may be multiple equilibrium allocations when $\mu > 0$. The following figure extracted from Azariadis and Kaas (2012) illustrates this result:

However, when the duration is infinite (i.e., when $\mu = 0$), $J(\cdot)$ takes the explicit formula

$$J(x) = \frac{\{(1-\beta\pi_L)\left[u(x) - u(\lambda)\right] + \beta(1-\pi_H)\left[u(c_L(x)) - u(e(L))\right]\}}{(1-\beta)(1+\beta-\beta(\pi_H+\pi_L))}$$

Azariadis and Kaas (2012) show that, when this is the case, there are at most one stationary trading equilibrium.

Azariadis and Kaas (2012) use the same non stochastic example with only one period of exclusion ($\mu = 1$) presented in the first version of their working paper (Azariadis and Kaas (2008)), to explain the intuition to why there may be multiple stationary equilibrium allocations in an economy with finite exclusion duration for defaulters. As before, their intuition relies on price complementarity (see section 4.2.1).

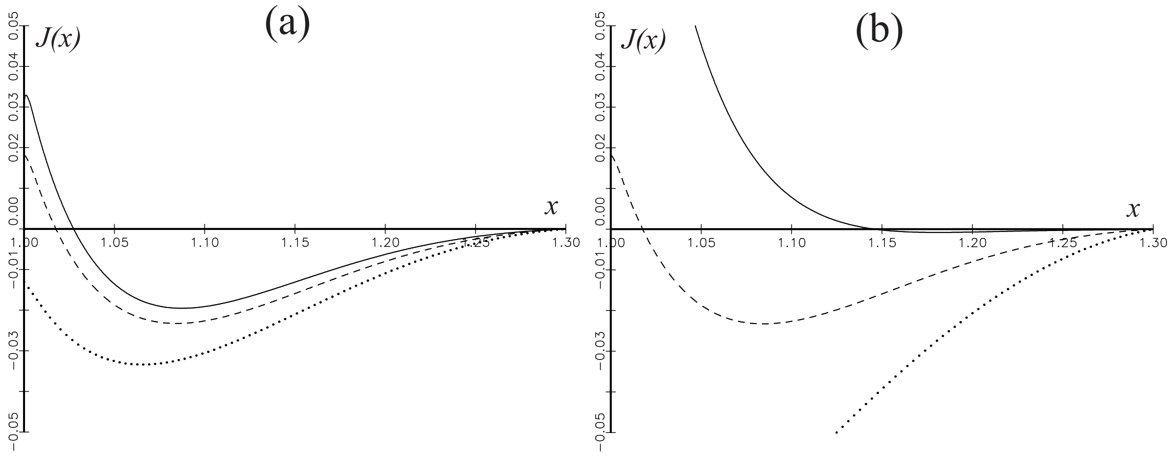


Figure 4.3: Curve $J(x)$ defining stationary equilibrium when $\mu = 1$, $\lambda = 1.3$, $u(c) = c^{1-\sigma}/(1-\sigma)$ and (a) $\sigma = 1.5$, $\beta = 0.5$ (dotted), $\beta = 0.7$ (dashed), $\beta = 0.9$ (solid) and (b) $\beta = 0.7$, $\sigma = 1.1$ (doted), $\sigma = 1.5$ (dashed) $\sigma = 1.9$ (solid).

## 4.4 Sequential equilibrium with permanent exclusion from borrowing (Hellwig and Lorenzoni (2009))

We are now going to present the model developed by Hellwig and Lorenzoni (2009). For this purpose, consider the environment described in section 2.2. Then we define a competitive equilibrium with self enforcing pricing debt as follows:

**Definition 4.4.1** *A competitive equilibrium with self-enforcing pricing debt consists in solvency constraints $\{\phi_i\}$, initial conditions $\{a_{i,0}\}$, quantities $\{c_i, a_i\}$ and prices $\{q\}$ such that:*

*i) Given $\{q\}$, for each $i$, $\{c_i, a_i\}$ solves*

$$V_{i,t}(a, z^t) = \max_{c,\{a_{z'}\}_{z' \in Z}} \left\{ u(c) + \beta \sum_{z' \in Z} V_{i,t+1}(a_{z'}, (z^t, z'))\pi(z', z_t) \right\}, \qquad (4.11)$$

*subject to*

$$e_{i,t}(z^t) + a = \sum_{z' \in Z} a_{z'} q_t(z^t, z') + c, \qquad (4.12)$$

$$a_{z'} \geq \phi_{i,t+1}(z^t, z') \quad \forall z' \in Z \qquad (4.13)$$

*ii) markets clear:*

$$\sum_i c_{i,t}(z^t) = \sum_i e_{i,t}(z^t) \quad \forall t, \, \forall z^t$$

$$\sum_{i \in \mathbf{I}} a_{i,t+1}(z^t, z') = 0 \quad \forall t, \, \forall z^t, \, \forall z'$$

*iii) Solvency constraints are not too tight:*

$$V_i(\phi_{i,t}(z^t), z^t) = V_{i,t}^D(0, z^t), \tag{4.14}$$

*where*

$$V_{i,t}^D(0, z^t) = \max_{c, \{a_{z'}\}_{z' \in Z}} \left\{ u(c) + \beta \sum_{z' \in Z} V_{i,t+1}(a_{z'}, (z^t, z'))\pi(z', z_t) \right\},$$

*subject to*

$$e_{i,t}(z^t) + a = \sum_{z' \in Z} a_{z'} q_t(z^t, z') + c,$$
$$a_{z'} \geq 0 \quad \forall z' \in Z \tag{4.15}$$

Notice that the participation constraint $V_{i,t}^D(0, z^t)$ depends on the asset prices $\{q_{i,t}\}$, which are endogenous.

### 4.4.1 Stationary example

Consider the case of only two agents, i.e., $I = \{0, 1\}$. In each period there are two possible states of nature: $\{s_0, s_1\}$. Incomes are given by

$$e_{0,t}(z_t = s_0) = \overline{e}, \quad e_{0,t}(z_t = s_1) = \underline{e} \quad \forall t$$
$$e_{1,t}(z_t = s_0) = \underline{e}, \quad e_{0,t}(z_t = s_1) = \overline{e} \quad \forall t,$$

where aggregate income is normalized to one, i.e., $\overline{e} + \underline{e} = 1$.

Suppose that $Pr(z_{t+1} = s_0 | z_t = s_1) = Pr(z_{t+1} = s_1 | z_t = s_0) \equiv \alpha \in (0, 1)$, $z_0 = s_0$ and that $a_0(z_0) = -\omega$ and $a_1(z_0) = \omega$.

**Proposition 4.4.1** *(Hellwig and Lorenzoni (2009), page 1144) Let $\overline{c}$ be the (unique) solution to the equation $1 - \beta(1-\alpha) = \beta\alpha u'(1-\overline{c})/u'(\overline{c})$. If $\overline{c} < \overline{e}$, there exists a stationary equilibrium with self enforcing private debt in which the following conditions are satisfied:*

*i) Arrow prices are $q(z^{t+1}) = q_c \equiv 1 - \beta(1-\alpha)$ if $z_{t+1} \neq z_t$ and $q(z^{t+1}) = q_{nc} \equiv \beta(1-\alpha)$ if $z_{t+1} = z_t$.*[6]

*ii) Consumption allocations are given by*

$$c_{i,t}(z^t) = \begin{cases} \overline{c}, & \text{if} \quad z_t = s_i \\ \underline{c}, & \text{if} \quad z_t \neq s_i \end{cases},$$

*where $\underline{c} = 1 - \overline{c}$.*

[6] Following the notation of Hellwig and Lorenzoni (2009), $q_{nc}$ stands for prices of assets that pay in the next period whenever there is no change in the current state of nature ($z_t = z_{t+1}$), whereas $q_c$ is the price of assets that pay in the next period, whenever the current state of nature changes ($z_t \neq z_{t+1}$).

*iii) Asset holdings are given by*

$$a_{i,t}(z^t) = \begin{cases} -\omega, & \textit{if} \quad z_t = s_i \\ \omega, & \textit{if} \quad z_t \neq s_i \end{cases},$$

*where* $\omega = (\overline{e} - \overline{c})/2q_c$.

*iv) Debt limits are* $\phi_{i,t}(z^t) = -\omega \ \forall \, z^t \succeq z_0$.

**Proof:** See the Appendix. ■

Notice that in equilibrium we must necessarily have $\overline{c} \leq \overline{e}$, otherwise the Euler equation wouldn't be satisfied. Indeed, suppose by contradiction $\overline{c} > \overline{e}$. Then, high income agents are constrained and low income are unconstrained, which implies, by the Euler equation, that

$$\begin{aligned} q_{nc} &= \beta \frac{\alpha u'(\overline{c})}{u'(1-\overline{c})} \\ &\geq \beta\alpha \frac{\alpha u'(1-\overline{c})}{u'(\overline{c})} \\ &\Rightarrow u'(\overline{c}) \geq u'(1-\overline{c}) \iff \overline{c} \leq 1-\overline{c} \leq 1-\overline{e} = \underline{e} \leq \overline{e} \rightarrow\leftarrow \end{aligned} \tag{4.16}$$

a contradiction.

Notice also that this equilibrium with binding solvency constraints requires that $q_c + q_{nc} = 1$, i.e., interest rate must be zero. Indeed let $c_h$ and $c_l$ be the stationary levels of consumption of an individual at high and low endowment periods, respectively. Then, the expected utility of an individual at a high endowment period is given by

$$v(c_h, c_l) = \frac{1}{1-\beta+2\beta\alpha}\left((1-\beta(1-\alpha))u(c_h) + \beta\alpha u(c_l)\right). \tag{4.17}$$

Suppose this consumer holds $-w$ assets at high income periods and $a \geq -\omega$ at low income periods. Then, the following budget constraints must be satisfied:

$$\begin{cases} c_h = \overline{e} - \omega + q_{nc}\omega - q_c a \\ c_l = \underline{e} + a - q_{nc}a + q_c\omega \end{cases}$$
$$\iff$$
$$(1-q_{nc})c_h + q_c c_l = (1-q_{nc})\overline{e} + q_c\underline{e} + [q_c^2 - (1-q_{nc})^2]w. \tag{4.18}$$

Therefore, the stationary consumption of an individual who stays solvent must maximize the value function 4.17 subject to 4.18.

If, on the other hand, the individual choses to default during a high income period, he will not hold positive levels of assets during high endowment periods, i.e., $\omega = 0$ ($\omega \neq 0$, would imply a contradiction). Therefore, a defaulting agent maximizes 4.17 subject to

$$(1-q_{nc})c_h + q_c c_l = (1-q_{nc})\overline{e} + q_c\underline{e}. \tag{4.19}$$

Now notice that 4.19 represents a shift on the budget constraint 4.18. This shift will be positive or negative depending on whether $[q_c^2 - (1 - q_{nc})^2]$ is positive or negative. If the shift is strictly positive, the agent could be made strictly better off by defaulting, whereas a negative shift implies the agent will prefer to stay solvent. Therefore, we have the following results:

- When $q_c + q_{nc} > 1$, the high income agent will have incentives to default;
- When $q_c + q_{nc} = 1$, the high income agent will be exactly indifferent between staying solvent and defaulting;
- When $q_c + q_{nc} < 1$, the high income agent will prefer to stay solvent.

Hence, in order to implement an stationary equilibrium with positive levels of debt in this simplified example, implied interest rates must be low in the Alvarez and Jermman (2000) sense (notice that zero interest rate implies low implied interest rates). In the next section, we will see that the requirement of low implied interest rate is also necessary to guarantee the existence of equilibrium with sustaining positive debt limits for the general case.

### 4.4.2 General case

The next proposition, gives the necessary and sufficient condition for the solvency constraints in problem 4.11 to be not too tight:

**Proposition 4.4.2** *(Hellwig and Lorenzoni (2009), page 1144) The debt limits $\{\phi_i\}$ are not too tight if and only if they allow for exact roll over:*

$$\phi_{i,t}(z^t) = \sum_{z^{t+1} \succeq z^t} q(z^{t+1})\phi(z^{t+1}) \quad \forall z^t \in Z^t.$$

By the next theorem, we can see why the requirement of low implied interest rate is a necessary condition to guarantee the existence of equilibrium with sustaining positive debt limits:

**Proposition 4.4.3** *(Bullow and Rogoff) Suppose prices and endowments are such that*

*i) Prices satisfy the high implied interest rates condition:*

$$\omega(z^t) \equiv \sum_{t=0} \sum_{z^{t+\tau} \succeq z^t} e_{i,t}(z^{t+\tau}) Q_0(z^{t+\tau}|z_0)/Q_0(z^t|z_0) < \infty \quad \forall t,\ z^t.$$

*ii) Solvency constraint rule out Ponzi schemes:*

$$\phi_{i,t}(z^t) \geq -\omega(z^t) \quad \forall t,\ z^t.$$

*Then, we must have $\phi_{i,t}(z^t) = 0 \quad \forall t,\ z^t$.*

Therefore, to guarantee the existence of equilibrium with positive debt, one of the conditions of proposition 4.4.3 must be violated. Hellwig and Lorenzoni (2009) proves the existence of equilibrium with sustaining positive debt by proving the equivalence between the allocations of the competitive equilibrium described above and the equilibrium with unbacked public debt.

# Appendix C

**Proof of proposition 4.2.1:**

(⇒) Suppose $b_t = B_t$, then by 4.5 and 4.6, we have that

$$\beta\frac{u'(1+\theta-x_{t+1})}{u'(x_t)} > \beta\frac{u'(x_{t+1})}{u'(1+\theta-x_t)}$$
$$\iff u'(1+\theta-x_{t+1})u'(1+\theta-x_t) > u'(x_t)u'(x_{t+1}) \tag{C.1}$$

Suppose by contradiction that $x_t + x_{t+1} \leq 1+\theta$. Then,

$$\begin{aligned}
&1+\theta-x_{t+1} \geq x_t \quad \text{and} \quad 1+\theta-x_t \geq x_{t+1}\\
&\iff u'(1+\theta-x_{t+1}) \leq u'(x_t) \quad \text{and} \quad u'(1+\theta-x_t) \leq u'(x_{t+1})\\
&\Rightarrow u'(1+\theta-x_{t+1})u'(1+\theta-x_t) \leq u'(x_t)u'(x_{t+1}),
\end{aligned}$$

a contradiction with C.1.

(⇐) Suppose $x_t + x_{t+1} > 1+\theta$, then

$$\begin{aligned}
&1+\theta-x_{t+1} < x_t \quad \text{and} \quad 1+\theta-x_t < x_{t+1}\\
&\iff u'(1+\theta-x_{t+1}) > u'(x_t) \quad \text{and} \quad u'(1+\theta-x_t) > u'(x_{t+1})\\
&\Rightarrow u'(1+\theta-x_{t+1})u'(1+\theta-x_t) > u'(x_t)u'(x_{t+1})\\
&\iff \beta\frac{u'(1+\theta-x_{t+1})}{u'(x_t)} > \beta\frac{u'(x_{t+1})}{u'(1+\theta-x_t)}\\
&\iff q_t > \beta\frac{u'(x_{t+1})}{u'(1+\theta-x_t)}\\
&\iff b_t = B_t.
\end{aligned}$$

■

**Proof of proposition 4.4.1:**

i) ⊢: The allocations satisfy the Euler Equation Clearly,

$$q_c = \beta\alpha\frac{u'(1-\bar{c})}{u'(\bar{c})}$$

satisfy the Euler equation, since the individual with high income must be unconstrained, which in turn implies the asset price must equal his MRS.

Equivalently,

$$q_{nc} = \beta(1-\alpha).$$

satisfy the Euler equation, since in a steady state equilibrium we must have

$$\begin{aligned} q_{nc} &= \max\left\{\beta(1-\alpha)\frac{u'(\overline{c})}{u'(\overline{c})}, \beta(1-\alpha)\frac{u'(1-\overline{c})}{u'(1-\overline{c})}\right\} \\ &= \beta(1-\alpha). \end{aligned}$$

ii) Market clearing is trivially satisfied.

iii) $\vdash$: Budget constraints are satisfied.

By the definition of $\omega$, we have that

$$\begin{aligned} \omega &= \frac{\overline{e}-\overline{c}}{2q_c} \\ 2q_c\omega &= \overline{e}-\overline{c} \\ 2(1-\beta(1-\alpha))\omega + \overline{c} &= \overline{e} \\ \omega q_c + \omega + q_{nc}(-\omega) + \overline{c} &= \overline{e} \\ q_c\omega + q_{nc}(-\omega) + \overline{c} &= \overline{e} - \omega \\ \iff q_c a_{0,t}(z_t = s_1) + q_{nc} a_{0,t}(z_t = s_0) + \overline{c} &= \overline{e} - a_{0,t}(z_t = s_0) \end{aligned}$$

Similarly, we have that the budget constraint of the individual with low income is satisfied.

iv) $\vdash$: Solvency constraints are not too tight.

From theorem 4.4.2, since $\phi(z^t)$ is constant, we must have $q_c + q_{nc} = 1$, which is satisfied.

■

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