Updating Pricing Rules *

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Abstract

This paper studies the problem of updating the super-replication prices of an arbitrage-free market in a multiperiod setting. We introduce a set of standard properties and a (*weak*) version of Dynamic Consistency to characterize the updated pricing rules by the Full Bayesian Rule. Since different pricing rules are related to different kinds of frictions on the financial markets, this study allow us to analyze the evolution of the market structure when new informations are revealed.

We also provide a geometric characterization for the pricing rules that characterizes frictionless incomplete markets. This geometric property is useful

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to demonstrate that the incomplete frictionless market structure is *invariant* under updating when a *non trivial updating condition* between the set of risk-neutral measures and revealed information is present.

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JEL Classification: D52 · D53

1 Introduction

When markets are complete and there are no frictions, a well-known result provided by Cox and Ross (Ross (1976, 1978) and Cox and Ross (1976)) shows that the cost of replication of any security is given by the mathematical expectation of its payoffs stream under the *unique* state contingent price or *risk-neutral* probability obtained by the no-arbitrage principle. That is, under this probability, the price process for any asset is a *martingale*. Moreover, this is equivalent to the existence of an *unique* stochastic discount factor.

This result is known in the literature as The Fundamental Theorem of Asset Pricing and it is considered a cornerstone of the modern theory of Mathematical Finance. Its main hypotheses relies on the lack of arbitrage opportunities and in the absence of market frictions such as bid-ask spreads and transactions costs. On the other hand, a growing empirical literature in Finance shows that the complete market assumption is an exception in real markets and that frictions and transactions costs plays an important role in the asset pricing. For instance, Luttmer (1996) used the New York Stock Exchange (NYSE) data to indicate that, taking transactions costs into accont, it is possible to make the low variability of the intertemporal marginal rates of substitutions of typical representative agent models consistent with asset returns.

In order to incorporate some of these frictions in the prices, the seminal work of Jouini and Kallal (1995) proposed a model of sublinear pricing for markets with no-arbitrage opportunities and in the presence of bid-ask spreads or incompleteness. Their main result shows that, under these hypotheses, the super-replication pricing rule of a given asset can be represented as the maximum of all expected values of its payoffs under a *set* of risk-neutral measures.

Chateauneuf et al. (1996) characterized Choquet pricing rules assuming the comonotonic additivity of the price functional. However, this axiom was criticized by Cerreia-Vioglio et al. (2015) with the argument that it is a dificult property to test. They propose an alternative set of axioms based also in the no-arbitrage but, instead assume the comonotone additivity property, they assume the Put-Call Parity for European options as the main property that ensures a Choquet valuation in a context of complete markets.

Araujo et al. (2012) main result characterizes the pricing rules that are associated to frictionless incomplete markets. Also, they show that when a Choquet pricing rule emerges from a frictionless incomplete market, the assets of this market form a set of bets in a partion of the state space.

In this work, we are interested in analyze what happens with the market structure when a new information is revealed, assuming that the information is incorporated using the concept of *Full Bayesian Update* (that consists in update all priors). This rule is also present in the works of Fagin and Halpern (1990), Jaffray (1992) and Pires (2002). More recently, the papers of Faro and Lefort (2013) and Galanis (2014) has studied weak forms of *dynamic consistency* under the Full Bayesian Update.

Our exposition is organized as follows: In the Section 2, we recall some definitons and some well-known results present in the literature and give a new geometric characterization for the set of risk-neutral measures of a frictionless incomplete market. Section 3 provides a study of the invariant market structures under the Full Bayesian Update. Section 4 concludes the paper and the proofs of the main results presented along this work are collected in the Appendix.

2 Framework

The purpose of this first section is revisit the main definitions and results about financial pricing rules which are super-replication rules of an arbitrage free market with a frictionless bond. Our contibution that appears in this section provides a novel geometric characterization for pricing rules of frictionless incomplete markets.

In the present work, we consider a three-period economy where the uncertainty is modeled by a finite state space set $S = \{s_1, s_2, ..., s_n\}$ and $t = 0, 1, 2$ represent the different time stages. An asset is a mapping $X : S \to \mathbb{R}$ that ensures the payment of $X(s)$ units of wealth in each state of nature $s \in S$ reveled in the period $t = 2$. Let Σ be the family of all subsets of S. Let Δ be the set of all probability functions $P : \Sigma \to [0, 1]$ and let Δ^+ be the set of all strictly positive probability functions, i.e, $P \in \Delta^+$ when $P(s) > 0$ for all $s \in S$. For any probability $P \in \Delta$ let $supp[P] = \{s \in S \mid P(s)\}$ be the support of P. Given an event $E \in \Sigma$ and a probability $P \in \Delta$, we say that P has full support on E if $supp[P] \subseteq E$.

In the initial period $t = 0$, the pricing rule describes the cost of to assume a position $X \in \mathbb{R}^S$. At the interim stage $t = 1$, the uncertainty is partially solved and the information whether the true state of nature is in a subset $E \in S$ or not is announced. At the final stage $t = 2$, the uncertainty is fully revealed and payment promises are fulfilled.

Definition 1 *A financial pricing rule* $C : \mathbb{R}^S \to \mathbb{R}$ *is a function over future payoffs contingenty to state space* $S = \{s_1, s_2, ..., s_n\}$ *in a three period context satisfying the following assumptions:*

(i) C is sublinear, i.e.,

$$C(\lambda X) = \lambda C(X), \text{ and}$$
$$C(X+Y) \leq C(X) + C(Y),$$

for all $X, Y \in \mathbb{R}^S$ and all non-negative real number λ;

(ii) C is arbitrage free, i.e., $C(X) > 0$ for any nonzero security $X \geq 0$;

(iii) C is normalized, i.e., $C(S^*) = 1$;

(iv) C is monotonic, i.e., $C(X) \geq C(Y)$ for all $X, Y \in \mathbb{R}^S$ s.t. $X \geq Y$;

(v) C is constant additive, i.e.,

$$C(X + kS^*) = C(X) + k,$$

for all $X \in \mathbb{R}^S$ and every real number k.

The following theorem that can be derived from Huber (1981), provides a dual characterization for every financial pricing rule:

Theorem 1 (Huber) *For any pricing rule satisfying conditions (i-v) there is a closed and convex set $\mathcal{K}$ of probability measures, where at least one element is strictly positive, such that for any security X*

$$C(X) = \max_{P \in \mathcal{K}} E_P(X)$$

A main example of pricing rule is the super-replication price of a securities market $\mathcal{M} = \{(X_j, q_j^A, q_j^B); 0 \leq j \leq m\}$, where $X_j \in \mathbb{R}^S$ are the tradeable assets with respective ask price q_j^A and bid price q_j^B. Then, the super-replication price is given by

$$C(X) = \inf \left\{ \sum_j \theta_j q_j^A - \sum_j \phi_j q_j^B \mid (\theta, \phi) \in \mathbb{R}_+^{2(m+1)} \wedge \sum_j (\theta_j - \phi_j) X_j \geq X \right\},$$

where θ_j denotes the number of units of asset j bought and ϕ_j denotes the number of units of asset j sold.

Observe that the above function is well-defined for all markets that offers no-arbitrage opportunities and one of its tradeble assets is the frictionless normalized riskless bond. That is, $X_0 = S^* := (1, 1, ..., 1)$ and $q_0^A = q_0^B = 1$. Furthermore, under these hypotheses, the result provided by Jouini and Kallal (1995) reports that the super-replication price can be represented as

$$C(X) = \max_{P \in \overline{\mathcal{Q}}} E_P(X),$$

where $\overline{\mathcal{Q}}$ is the closure of the set of risk-neutral probabilities defined as:

$$\mathcal{Q} := \left\{P \in \Delta^+ \mid q_j^A \leq E_P(X_j) \leq q_j^B, \forall 0 \leq j \leq m\right\}.$$

Hence, every super-replication price of a securities market with no-arbitrage opportunities is a pricing rule as mentioned in Definition 1. However, the converse is not always true. As noticed by Araujo et al. (2015), using the Theorem 2.4.6 of Schneider (1993) it is possible to demonstrate that the closure of the set of risk-neutral probabilities characterizes a market with a finite number of securities if and only if this set is a polytope[1]. They also provide a characterization interim of pricing rules of "finitely generated pricing rules".

For every pricing rule, Araujo et al. (2012) introduced the following sets:

$$F_C := \left\{X \in \mathbb{R}^S \mid C(X) + C(-X) = 0\right\};$$

$$L_C := \left\{X \in \mathbb{R}^S \mid \forall Y \in \mathbb{R}^S \text{ s.t. } Y > X,\ C(Y) > C(X)\right\};$$

$$\mathcal{Q}_C := \{P \in \Delta \mid E_P(X) = C(X),\ \forall X \in F_C\}.$$

The first is the set of all frictionless securities. That is, it is the set of all assets that can be bougth and sold by the same price. The second is the set of all undominated securities. And the third is the set of all probabilities that agree about the price of every frictionless securities.

The main result in Araujo et al. (2012) is a complete characterization of those pricing rules $C(\cdot)$ that are super-replication prices of a frictionless and arbitrage-free incomplete market with a frictionless bond. Their characterization relies on the equivalence between the set of frictionless securities and undominated securities defined by $C(\cdot)$. More precisely:

Theorem 2 (Araujo, Chateauneuf and Faro) *A pricing rule C is a super- replication price of a frictionless and arbitrage-free complete or incomplete market of tradeable securities including the riskless bond if, and only if, C is a financial pricing rule satisfying $F_C = L_C$.*

In the oncoming result, we characterize this same market structure looking to the geometric properties of the set of risk-neutral measures $\mathcal{K}$. In order to establish that result, we first present the following definition:

Definition 2 *A set $\mathcal{P} \subset \Delta$ is called* ***non-expansible*** *if*

$$\{\alpha P + (1-\alpha)Q \mid P, Q \in \mathcal{P}, \alpha \in \mathbb{R}\} \cap \Delta = \mathcal{P}.$$

[1]A set $\mathcal{P} \subset \Delta$ is called a *polytope* if there exist $P_1, ..., P_k \in \Delta$ such that $\mathcal{P} = \text{conv}\{P_1, ..., P_k\}$.

We assume that the empty set is non-expansible. Observe that the only non-expansible sets $\mathcal{P}$ such that $\mathcal{P} \subset \Delta^+$ are the singletons. Futhermore, any non-expansible set with at least two points have its extremal points over the frontier of the simplex Δ. Nonetheless, that is not a sufficient condition to determine whether a polytope is or not expansible. As illustrated in Figure 1, the left set is expansible since its extremal points (A and B) are strictly positive probabilities and the set in the right which its extremal points (C and D) are on the frontier of Δ is a non-expansible set. On the other hand, the extremal points of the polytope $\mathcal{P} = \text{conv}\{(0.5, 0.5, 0), (0.5, 0, 0.5), (0, 0.5, 0.5)\}$ lies on the frontier of Δ but $\mathcal{P}$ is a expansible set. In fact, observe that $P = (0.5, 0.25, 0.25)$ and $Q = (0, 0.5, 0.5)$ are two points of $\mathcal{P}$ and for $\alpha = 1.5$ we have that $\alpha P + (1 - \alpha)Q = (0.75, 0.125, 0.125)$ is a positive probability which not lies on $\mathcal{P}$ (see Figure 2). This latter example was also analysed by Araujo et al. (2015) and they showed that the underlying market for this set of risk-neutral measures has frictions over tradeable securities.

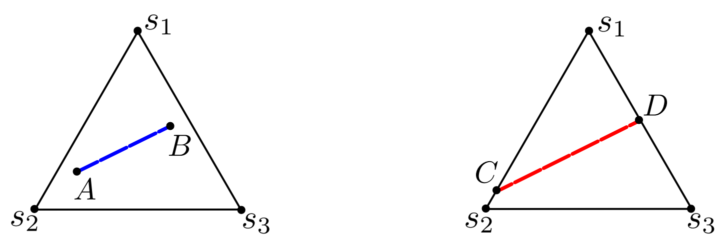

Figure 1: Example of an expansible set (left) and its "expanded version" (right).

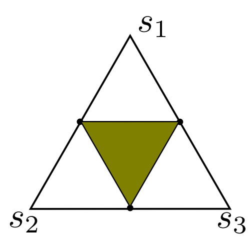

Figure 2: The polytope $\mathcal{P}$ is expansible.

Now, we present our first result about pricing rules. It is a complete geometric characterization for any pricing rule which is super-replication rule of a frictionless and arbitrage free market with a frictionless bond.

Theorem 3 *Let $\mathcal{K}$ be a non-expansible polytope with at least one interior point, then*

$$C(X) := \max_{P \in \mathcal{K}} E_P(X),$$

satisfy $L_C = F_C$. *Also, if* C *is a pricing rule satisfying* $F_C = L_C$*, then* $\mathcal{K}$ *is a non-expansible polytope with at least one interior point.*

Notice that the geometric property of non-expansibility of the set of risk-neutral measures could appears more naturally to the empirical finance literature than the analytical equivalence between L_C and F_C. Also, could be hard to highlight the set L_C of all undominated securities since this property requires a contingent claim representation for the assets considered. Despite of natural errors in collected data, it could be worthwhile use empirical analysis in order to distinguish the markets which are incomplete and frictionless from those are not.

Theorem 4 *Let* $\mathcal{K}$ *a polytope with a finite number of extremal points given by* $\{Q_1, ..., Q_n\}$. *Define the set*

$$\widetilde{\mathcal{K}} := \left\{ \sum_{i=1}^{n} \alpha_i Q_i \mid \alpha_i \in \mathbb{R} \text{ and } \sum_{i=1}^{n} \alpha_i = 1 \right\} \cap \Delta.$$

Then, $\widetilde{\mathcal{K}}$ *is the smallest (by inclusion) non-expansible set which contains* $\mathcal{K}$. *That is,*

$$\widetilde{\mathcal{K}} = \bigcap_{\mathcal{L} \supseteq \mathcal{K}} \{\mathcal{L} \mid \mathcal{L} \text{ is expansible}\}.$$

Suppose a situation where we have a pricing rule C with set of risk-neutral measures given by a expansible set $\mathcal{K}$ and we do not have to much information about it. Next, we find a result with a clear message about $\widetilde{\mathcal{K}}$. Is it possible to tell something about the original pricing rule C?

Proposition 1 *Let* $C(X) = \max_{P \in \mathcal{K}} E_P(X)$ *the super-replication pricing rule of a given arbitrage-free financial market* $\mathcal{M}$. *Let* $\widetilde{\mathcal{K}}$ *be the smallest non-expansible set which contains* $\mathcal{K}$. *Then, the pricing rule* $\widetilde{C}(X) := \max_{P \in \widetilde{\mathcal{K}}} E_P(X)$ *is the super-replication pricing rule of a frictionless and arbitrage-free financial market* $\widetilde{\mathcal{M}}$ *such that*

$$F_C = F_{\widetilde{C}}.$$

That is, $\mathcal{M}$ *and* $\widetilde{\mathcal{M}}$ *have the same set of frictionless securities.*

3 Updating Pricing Rules

Suppose that in some point of the time between the purchases and the securities liquidation, a new information about the true state of nature is revealed. How this information impacts the asset prices? Is it possible a change in the market structure after this revelation? In this section, we propose a novel approach to characterize the updated pricing rules which satisfies the standard conditions proposed in the Definition 1 and the forthcoming property called *Dynamic Consistency to Certainty*.

Given an event $E \subseteq S$ and a fixed pricing rule C, we say that E is *relevant* if $-C(-E^*) > 0$. Notice that if E is relevant, then $P(E) > 0$ for all $P \in \mathcal{K}$, where $\mathcal{K}$ is the set of risk neutral measures which characterizes the pricing rule C. Let $\mathcal{R}$ be the set of all relevants events. Let us denote the updating pricing rule by $C^E(\cdot)$. In order to $C^E(\cdot)$ be qualified as a pricing rule[2], it is necessary that $C^E(\cdot)$ satisfies the five conditions present in Definition 1. Moreover, let us focus on the case where the set of risk-neutral measures is a polytope. Also, we impose a property which links the unconditional and the conditional pricing rules in a restrictive class of assets. For any asset X and real number k we define the asset XEk as $XEk(s) = X(s)$ when $s \in E$ and $XEk(s) = k$ when $s \in E^C$. Inspired in the axiomatic approach proposed by Pires (2002), we have the following definition:

Definition 3 *A given collection of pricing rules $\{C, C^E\}_{E\in\mathcal{R}}$ satisfies the Dynamic Consistency to Certainty (DCC) property if for any asset X and real number k,*

$$C(XEk) \geq k \text{ if and only if } C^E(X) \geq k.$$

In other words, given an event E and an asset $Y = XEk$ that is potential risky on E and riskless on E^C. That is, the asset Y delivery the same amount k in every state $s \in E^C$. If the unconditional price of Y is greater than or equal to k, then its conditional price must also be greater than or equal to k.

Since C^E is a pricing rule, there a set of risk neutral measure $\mathcal{L}$ such that $C^E(x) = \max_{P\in\mathcal{L}} E_P(X)$. Then, it is possible to ask which conditions ensures there is a relation between the sets $\mathcal{L}$ and $\mathcal{K}^E := \{P^E \in \Delta \mid P \in \mathcal{K}\}$, where P^E is the Bayesian Update of the probability P, given by $P^E(F) = P(F \cap E)/P(E)$ for every $F \subseteq S$. The next result shows that Full Bayes Rule is the unique pricing rule that satisfies the DCC property.

Theorem 5 *Let $C(\cdot)$ be a pricing rule characterized by set $\mathcal{K}$ and $\mathcal{R}$ be the set of relevant events. Then, the following conditions are equivalent:*

(i) The collection of pricing rules $\{C, C^E\}_{E\in\mathcal{R}}$ satisfies the DCC property.

(ii) The updated pricing rule is given by $C^E(X) := \max_{P\in\mathcal{K}^E} E_P(X)$, for all $E \in \mathcal{R}$.

[2]Observe that $C^E(\cdot)$ must be an application from $\mathbb{R}^E$ to $\mathbb{R}$. Then, $C^E(X)$ means $C^E(\tilde{X})$, where $\tilde{X} \in \mathbb{R}^E$ is the restriction of X under E.

This Theorem states that if the DCC property is satisfied and $P(E) > 0$ for all $P \in \mathcal{K}$, then the set of risk neutral measures should be updated by the Full Bayesian Rule. The idea of update all probabilities of a given set was first proposed by Fagin and Halpern (1990) and Jaffray (1992). Pires (2002) and Faro and Lefort (2013) provided a decision-theoretic axiomatization of the Full Bayesian Rule in different frameworks. We are unware of any other work which was provided an axiomatization of updating pricing rules.

In the remainder of this section we analyze what happens with different market structures when a new information is revealed. Henceforward, we always assume that $\mathcal{K} \subseteq \Delta$ is a set of probabilities, $E \in \Sigma$ an event such that $P(E) > 0$ for all $P \in \mathcal{K}$ and that $\mathcal{K}^E = \{P^E \in \Delta \mid P \in \mathcal{K}\}$ is the set of conditional probabilities P^E.

3.1 Updating Incomplete Frictionless Markets

An important class of pricing rules are those ones that represents a super-replication price of an incomplete frictionless market. As commented early, these pricing rules are characterized by non-expansible sets. Therefore, a natural question arises: Is the Full Bayesian Update of a non-expansible set, also non-expansible? Unfortunately, this is not true in general. As shown in the next example, the update of a non-expansible set can be expansible.

Example 1 *Consider the pricing rule $C : \mathbb{R}^3 \to \mathbb{R}$ defined by,*

$$C(X) = \max\{E_P(X), E_Q(X)\},$$

where $P = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$ and $Q = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$. Note that the underlying set of risk-neutral probabilities is given by $\mathcal{K} = \{\alpha P + (1-\alpha)Q \mid \alpha \in [0,1]\}$. If the event $E = \{s_1, s_2\}$ is revealed, the Bayesian update of the above set is

$$\mathcal{K}^E = conv\left\{(1,0), \left(\frac{1}{2}, \frac{1}{2}\right)\right\}.$$

This example shows that, starting in a situation where the financial market is frictionless but incomplete, it is possible to obtain bid-ask spreads performing updates given by the Full Bayesian Rule. Therefore, if the unconditional set of risk neutral measures is non-expansible, we need to impose an additional hypothesis under this set in order to ensure that its Bayesian update is also a non-expansible set.

Definition 4 *Let $\Delta(E)^+$ be the set of strictly positive probabilities over E. We say that $\mathcal{K}^E$ is a* ***non trivial updating*** *of $\mathcal{K}$ under the relevant event E if for every $P \in \mathcal{K}^E \cap \Delta(E)^+$ there is a probability $Q \in \mathcal{K}$ such that $Q(s) > 0$ for all $s \in E^C$ and $Q^E = P$.*

Observe that, when the non trivial updating condition is present, if there is $P \in \mathcal{K} \cap \Delta(E)^+$, then $P^E = P$. The existence of another probability $Q \neq P$ such that $Q^E = P^E$ can be interpreted as an ambiguity to choose the "correct" prior. The next result shows that the incomplete frictionless market structure is *invariant* under updating when a non trivial updating condition is present.

Theorem 6 *If $\mathcal{K}$ is the set of risk-neutral measures of a frictionless complete or incomplete finite market and $E \in \Sigma$ an event such that $P(E) > 0$ for all $P \in \mathcal{K}$, then are equivalent:*

(i) $\mathcal{K}^E$ is a non trivial updating of $\mathcal{K}$;

(ii) $\mathcal{K}^E$ is the set of risk-neutral measures of a frictionless incomplete finite market;

Backing to Example 1, notice that non trivial updating condition is not present. Indeed, $P = \left(\frac{1}{2}, \frac{1}{2}, 0\right) \in \mathcal{K} \cap \Delta(E)^+$ but there is no other probability $Q \in \mathcal{K}$ such that $Q \neq P$ and $Q^E = P$. Observe that, although $\mathcal{K}$ to be a non-expansible set characterizing a frictionless incomplete market, the set $\mathcal{K}^E$ is expansible and related with a market with bid-ask spreads (see Figure 3).

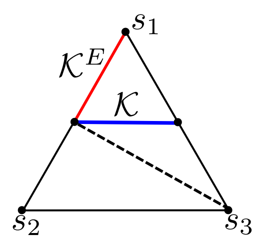

Figure 3: When the non trivial updating condition is not present.

3.2 Updating Markets with Uniform Bid-Ask Spreads

Now, we provide a result about the invariance of market's structure when frictions are present. More specifically, we analyze what happens to a complete market with a frictionless bond and uniform bid-ask when a new information is revealed. We define this class of market in the same way as given by Araujo et al. (2015):

Definition 5 *We say that $\mathcal{M} = \left\{X_j; (q_j^A, q_j^B)\right\}_{j=0}^m$ is market with a frictionless bond and uniform bid-ask spreads if:*

i) $X_0 = S^*$, $m = \#S$, and for all $j \in S$, $X_j = \{j\}^*$;

ii) For all $j \in S$, the bid-ask spread prices of $X_j = \{j\}^*$ is given by

$$q_j^A - q_j^B = \varepsilon \quad \text{and} \quad 1 - \sum_{j=1}^{m} q_j^B = \varepsilon.$$

Araujo et al. (2015) showed that all markets with a frictionless bond and uniform bid-ask can be characterized by a set of risk-neutral measures $\mathcal{K}$ such that $\mathcal{K} = (1-\varepsilon)Q + \varepsilon\Delta$ for some probability $Q \in \Delta^+$. In this case, the pricing rule is a convex combination between the "pure price" $E_Q(X)$ and the worst scenario payoff for the seller. That is,

$$C(X) = (1-\varepsilon)E_Q(X) + \varepsilon \max_{s\in S} X(s).$$

Furthermore, the bid-ask spread for any asset X is given by

$$BA(X) = \varepsilon(\max_{s\in S} X(s) - \min_{s\in S} X(s)).$$

Unlike the case of incomplete frictionless market, this result do not require any strong condition over the underlying market. Indeed, no extra condition is necessary to demonstrate the preservation of the market structure.

Theorem 7 *The structure of a market with a frictionless bond and uniform bid-ask spreads is always preserved by Bayesian update. Furthermore, the updated bid-ask spread is given by*

$$\gamma := \frac{\varepsilon}{(1-\varepsilon)\sum_{s'\in E} Q(s') + \varepsilon}.$$

As an immediate consequence of the previous result, observe that it is possible calculate the updated bid-ask spread of any asset X in terms of its payoff and the pure price Q. In fact, we have

$$BA^E(X) := \gamma(\max_{s\in E} X(s) - \min_{s\in E} X(s)).$$

Furthermore, we can focus on the bid-ask updating ratio given by

$$\sigma^E(X) := \frac{BA^E(X)}{BA(X)}.$$

When the sets $\{\arg\max_{s\in S} X(s)\} \cap E$ and $\{\arg\min_{s\in S} X(s)\} \cap E$ are non-empty, this ratio becomes

$$\sigma^E(X) = \left((1-\varepsilon)\sum_{s'\in E} Q(s') + \varepsilon\right)^{-1} \geq 1.$$

Then, when the market is dealing with some asset which its worst and best scenarios are not excluded after an early resolution of the uncertainty, the new information will increase the asset's friction, raising its bid-ask spread. However, the increase in the bid-ask spread does not occur for all assets.

Example 2 *Consider a complete market with pure price $Q = (0.25, 0.25, 0.25, 0.25)$ and uniform bid-ask spread given by $\varepsilon = 0.5$. Let $X = (5, 3, 2, 1)$ be a particular asset and let $E = \{s_2, s_3\}$ be an event revealed at $t = 1$. Then,*

$$BA(X) = 0.5(5-1) = 2 \quad \text{and} \quad BA^E(X) = \frac{0.5}{0.5(0.25+0.25)+0.5}(3-2) = 0.667.$$

Observe that the best and the worst payoffs of the asset X occurs in states which are outside the event E. Then, the large distortion that these payoffs cause in prices will no more be present when the event E is revealed, diminishing the bid-ask spread.

3.3 Updating Markets with Put-Call Parity

In this subsection we want to analyze if a new information can affect the Put-Call Parity (henceforth PCP) in a particular market. Given a asset $X \in \mathbb{R}^S$, we denote by K_q^X the call option on X with strike price $q \geq 0$ and P_q^X the put option with strike price $q \geq 0$. Since K_q^X is the contingent claim given by $(X - qS^*)^+$ and P_q^X is given by $(qS^* - X)^+$, we have following equality:

$$K_q^X - P_q^X = X - qS^*.$$

A pricing rule C satisfy the PCP if

$$C(K_q^X) + C(-P_q^X) = C(X) - q.$$

As a straightforward consequence of the main result presented by Cerreia-Vioglio et al. (2015), it is possible to show that all pricing rules which satisfies the PCP property are given by the Choquet's integral of a given concave capacity ν. Furthermore, the market's set of risk-neutral measures is the anticore of this capacity ν.[3]

The next example shows that the Put-Call Parity is not an invariant property when the market's fundamentals are updated through the Full Bayesian Rule.

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[3] A capacity ν is concave if:

$$\nu(A \cup B) + \nu(A \cap B) \leq \nu(A) + \nu(B), \forall A, B \subset S.$$

The *anticore* of the capacity ν is the closed, convex and bounded set:

$$\text{core}(\nu) = \{P \in \Delta : P(A) \leq \nu(A) \; \forall A \in \Sigma\}.$$

Example 3 *Consider $S = \{s_1, s_2, s_3, s_4\}$ and the following concave capacity ν such that*

$$\nu(s_1) = \nu(s_2) = \nu(s_4) = \frac{5}{8}, \quad \nu(s_3) = \frac{3}{8}$$
$$\nu(s_i \cup s_j) = \frac{6}{8} \text{ and } \nu(S - \{s_i\}) = \frac{7}{8}$$

The anticore of this capacity is given by $\mathcal{K} = \text{co}\{P_1, P_2, P_3, P_4, P_5, P_6\}$, where

$$P_1 = \left(\frac{5}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right); \quad P_2 = \left(\frac{1}{8}, \frac{5}{8}, \frac{1}{8}, \frac{1}{8}\right); \quad P_3 = \left(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{5}{8}\right);$$

$$P_4 = \left(\frac{3}{8}, \frac{1}{8}, \frac{3}{8}, \frac{1}{8}\right); \quad P_5 = \left(\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}\right); \quad P_6 = \left(\frac{1}{8}, \frac{1}{8}, \frac{3}{8}, \frac{3}{8}\right);$$

However, if we consider the event $E = \{s_2, s_3, s_4\}$, the updated set $\mathcal{K}^E$ of new risk-neutral measures will be given as the convex hull of P_2^E, P_3^E, P_4^E, P_5^E, P_6^E. In this case, $\mathcal{K}^E$ cannot be the anticore of a concave capacity μ. Otherwise, μ would be given by

$$\mu(A) = \max_{P \in \mathcal{K}^E} E_P(A)$$

and $P^* = (\frac{9}{35}, \frac{3}{5}, \frac{1}{7}) \in \Delta(s_2, s_3, s_4)$ is such that $P^* \in \text{acore}(\mu)$, but $P^* \notin \mathcal{K}^E$.

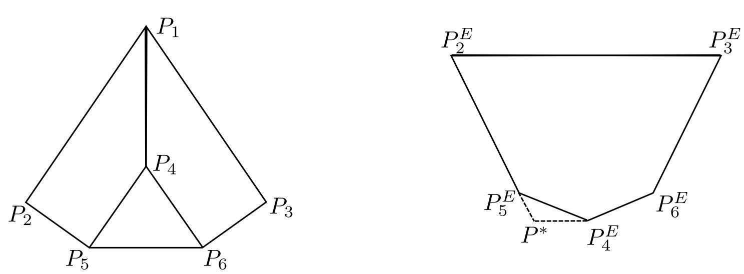

Figure 4: Geometric representation of the sets $\mathcal{K}$ and $\mathcal{K}^E$ presented in the Example 3.

A natural question that arises in this context is whether there is any direct relation between $C^E(X)$ and $C(X)$ or not. In general, it is not possible determine $C^E(X)$ in terms of $C(X)$ through a closed formula. However, when the set of risk-neutral measures $\mathcal{K}$ of a particular market is the anti-core of a regular concave capacity ν, a formal relation between $C^E(X)$ and $C(X)$ exists for all assets X.

If ν is a concave capacity and E is an event such that $\nu(E^C) < 1$, we define the *Full Bayesian Update* of ν given the event E as:

$$\nu^E(A) = \sup\{P^E(A) \mid P \in \text{acore}(\nu)\}, \forall A \subset E.$$

When ν is concave, Jaffray (1992) have showed that ν^E can be written as the following:

$$\nu^E(A) = \frac{\nu(A)}{\nu(A) + 1 - \nu(A \cup E^C)}. \tag{1}$$

A concave capacity ν is called *regular* if $\text{acore}(\nu^E) = (\text{acore}(\nu))^E$ and $\nu(E^C) < 1$ for every event $E \neq \emptyset$.

Theorem 8 *Let C be a pricing rule given by $C(X) = \max_{P \in \mathcal{K}} E_P(X)$. Then, the following assertions are equivalents:*

(i) For every event $E \neq \emptyset$, $C^E(X)$ is a Choquet integral.

(ii) The set $\mathcal{K}$ is the anti-core of a regular concave capacity ν.

(iii) For every event $E \neq \emptyset$, $C^E(X) = \int X d\nu^E$, where ν^E is defined as in (1) and $\nu := \nu^S$ is the unique capacity such that $\text{acore}(\nu) = \mathcal{K}$.

In other words, the Theorem 8 states that the PCP property is invariant through all possible non empty events E if and only if the set of risk-neutral measures is the anti-core of a **regular** concave capacity ν. This kind of capacity has been analysed in the work of Chateauneuf et al. (2011). Such capacities are concave strictly monotonic (ε, δ)-contaminations[4]. In particular, markets with uniform bid-ask spreads are generated by the so-called ε-contamination capacities (when $\delta = 0$). Therefore, from Proposition 3 in Chateauneuf et al. (2011) it is posible to see that the Theorem 8 is a generalization of the Theorem 7.

4 Conclusion

In this paper, we present a geometric characterization for the set of risk-neutral measures of an incomplete frictionless market. This technical finding allow us proof that the incomplete market structure is invariant under the Full Bayesian Update when the non trivial updating condition is present. Another important contribution is the axiomatization for the updating pricing rules satisfying a (weak) dynamic

[4] A capacity ν on (S, Σ) is a concave strictly monotonic (ε, δ)-contamination if there exists a probability P_0 strictly positive and $\varepsilon, \delta \in \mathbb{R}$ such that

$$\nu(A) = (1 - \varepsilon)P_0 + \varepsilon + \delta,$$

for all $A \neq \emptyset$ or $A \neq S$, $\nu(\emptyset) = 0$ and $\nu(S) = 1$, $\delta \in \left[0, \frac{\alpha}{1-\alpha}\right]$ and $\varepsilon \in \left[-\delta, 1 - \frac{\delta}{\alpha}\right]$, where $\alpha = \min_{s \in S} P_0(\{s\})$.

consistency property. We also provide a result in which the markets preserve the Put-Call Parity property after a partial resolution of the uncertainty.

However, some problems remain open. For instance, we do not know which conditions are necessary to maintain the set of risk-neutral measure as the anti-core of a non-regular concave capacity. This question is important since it is related to the Put-Call Parity. These issues will be our starting point in a future research about updating pricing rules.

5 Appendix

Proof of Theorem 3: Suppose that $X \notin F_C$ and let $Q_1, Q_2, ..., Q_n$ be all the extremal points of $\mathcal{K}$ such that $E_{Q_1}(X) = \cdots = E_{Q_n}(X) = C(X)$. That is,

$$\{Q_1, Q_2, ..., Q_n\} = Ext(\mathcal{K}) \cap \arg\max_{P \in \mathcal{K}} E_P(X).$$

Since $X \notin F_C$, there is $P \in \mathcal{K}$ such that $C(X) > E_P(X)$. Indeed, by Lemma 4 in Araujo et al. (2012), it is known that $X \in F_C$ if and only if $E_P(X) = E_Q(X)$ for all $P, Q \in \mathcal{K}$. Furthermore, by the Krein-Milman's Theorem[5], we can assume without loss of generality that P is also an extremal point of $\mathcal{K}$. Indeed, if $C(X) = E_P(X)$ for all $P \in Ext(\mathcal{K})$, then $C(X) = E_P(X)$ for all $P \in \mathcal{K}$, a contradiction.

Observe that there is a state s such that $Q_i(s) = 0$ for all $i = 1, ..., n$. Otherwise, the probability Q defined as $Q = \sum \frac{1}{n} Q_i$ is an interior point of the simplex Δ. Therefore, there will be a positive real ε such that $(\varepsilon + 1)Q - \varepsilon P \in \Delta$. Furthermore, since $\mathcal{K}$ is non-expansible, we have that $(\varepsilon + 1)Q - \varepsilon P \in \mathcal{K}$.

On the other hand,

$$E_{(\varepsilon+1)Q - \varepsilon P}(X) = E_Q(X) + \varepsilon(E_Q(X) - E_P(X)) > E_Q(X) = C(X).$$

And this cannot be true.

Now, if s is a state such that $Q_i(s) = 0$ for all $i = 1, ..., n$, define

$$Y = X + \delta\{s\}^*.$$

For any extremal point P such that $C(X) > E_P(X)$ we can choose $\delta > 0$ sufficiently small such that $C(X) > E_P(X + \delta\{s\}^*)$. Since the number of extremal points of $\mathcal{K}$ is finite, we can choose $\delta > 0$ such that $C(X) > E_P(X + \delta\{s\}^*)$ for all extremal point $P \notin \{Q_1, Q_2, ..., Q_n\}$.

Therefore, it is possible to choose $Y = X + \delta\{s\}^*$. such that $Y > X$ and $C(Y) = C(X)$. Hence, $X \notin L_C$.

[5]The Krein-Milman's Theorem says that any convex and compact set (in $\mathbb{R}^n$) is the convex hull of its extremal points (see Corollary 18.5.1, page 167 of Rockafellar (1997)).

Since $F_C \subset L_C$ for all pricing rules, the first part of the theorem follows.

The converse is also true: If C is a pricing rule such that $F_C = L_C$, by the Lemma 21 proved in Araujo et al. (2012), then

$$\mathcal{K} = \mathcal{Q}_C := \{P \in \Delta \mid E_P(X) = C(X), \forall X \in F_C\}.$$

Note that $\mathcal{Q}_C$ is a non-expansible set. Indeed, if $P_1, P_2 \in \mathcal{Q}_C$ and $\alpha \in \mathbb{R}$ such that $\alpha P_1 + (1-\alpha)P_2 \in \Delta$, it is immediate to see that we also have $\alpha P_1 + (1-\alpha)P_2 \in \mathcal{Q}_C$. Therefore, $\mathcal{K}$ is a non-expansible set. □

Proof of Theorem 4: We devide the proof in three small steps. First, by definition $\mathcal{K} \subset \widetilde{\mathcal{K}} \subset \Delta$. Now, observe that $\widetilde{\mathcal{K}}$ is a convex set. Indeed, let $P, Q \in \widetilde{\mathcal{K}}$ be two probabilities with

$$P = \sum_{i=1}^{n} \alpha_i Q_i \quad \text{and} \quad Q = \sum_{i=1}^{n} \beta_i Q_i,$$

where $\alpha_i, \beta_i \in \mathbb{R}$ with $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i = 1$. If $\theta \in [0,1]$, then

$$\theta P + (1-\theta)Q = \sum_{i=1}^{n} (\theta\alpha_i + (1-\theta)\beta_i) Q_i \in \widetilde{\mathcal{K}}.$$

Second, let us prove that $\widetilde{\mathcal{K}}$ is non-expansible. If $P, Q \in \widetilde{\mathcal{K}}$, there are $\alpha_i, \beta_i \in \mathbb{R}$ with $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i = 1$ such that

$$P = \sum_{i=1}^{n} \alpha_i Q_i \quad \text{and} \quad Q = \sum_{i=1}^{n} \beta_i Q_i,$$

Let $\theta \in \mathbb{R}$ be a parameter such that

$$\theta P + (1-\theta)Q = \sum_{i=1}^{n} (\theta\alpha_i + (1-\theta)\beta_i) Q_i \in \Delta.$$

Since $\sum_{i=1}^n (\theta\alpha_i + (1-\theta)\beta_i) = 1$, we have $\theta P + (1-\theta)Q \in \widetilde{\mathcal{K}}$. Therefore, $\widetilde{\mathcal{K}}$ is non-expansible. Finally, let C be a non-expansible set such that $\mathcal{K} \subset C$. Let us prove that $\widetilde{\mathcal{K}} \subset C$. Suppose that $R \in \widetilde{\mathcal{K}}$. By definition, there are $\alpha_i \in \mathbb{R}$ such that $\sum_{i=1}^n \alpha_i = 1$ and $R = \sum_{i=1}^n \alpha_i Q_i$. Assume without loss of generality that $\alpha_i > 0$ for all $1 \leq i \leq m$ and $\alpha_i \leq 0$ for all $m+1 \leq i \leq n$. Let $\alpha = \sum_{i=1}^n \alpha_i$ be the sum of all positive weights. Observe that R can be rewritten as

$$R = \alpha \sum_{i=1}^{m} \frac{\alpha_i}{\alpha} Q_i + (1-\alpha) \sum_{i=m}^{n} \frac{\alpha_i}{1-\alpha} Q_i.$$

Since the weights $\left(\frac{\alpha_i}{\alpha}\right)$, when $\alpha_i > 0$ and $\left(\frac{\alpha_i}{1-\alpha}\right)$, when $\alpha_i \leq 0$ are all positive, we must have $R \in C$. Therefore,

$$\widetilde{\mathcal{K}} = \bigcap_{\mathcal{K} \subseteq \mathcal{L}} \{\mathcal{L} \mid \mathcal{L} \text{ is expansible}\}.$$

□

Proof of Proposition 1: Given a pricing rule C and the corresponding set F_C, recall that $X \in F_C$ if and only if $E_P(X) = E_Q(X)$ for all $P, Q \in \mathcal{Q}$. Now, suppose that $X \in F_{\widetilde{C}}$. Then, $E_P(X) = E_Q(X)$ for all $P, Q \in \widetilde{\mathcal{K}}$. Since $\mathcal{K} \subset \widetilde{\mathcal{K}}$, we have $E_P(X) = E_Q(X)$ for all $P, Q \in \mathcal{K}$. Therefore, $X \in F_C$.

Now, if $X \in F_C$, we have $E_{Q_i}(X) = E_{Q_j}(X)$ for all extremal points $Q_i, Q_j \in \mathcal{K}$. Then, for every two vectors $\alpha \in \mathbb{R}^n$ and $\beta \in \mathbb{R}^m$ such that $\sum_{i=1}^n \alpha_i = \sum_{i=1}^m \beta_i = 1$ and $\sum_{i=1}^n \alpha_i Q_i \in \Delta$, $\sum_{i=1}^m \beta_i Q_i \in \Delta$, we have

$$E_{\sum_{i=1}^n \alpha_i Q_i}(X) = E_{\sum_{i=1}^m \beta_i Q_i}(X).$$

Therefore, $X \in F_{\widetilde{C}}$. □

Proof of Theorem 5: $(ii) \Rightarrow (i)$. It is straightforward to see that $C^E(\cdot)$ given by the Full Bayesian Update rule fulfills all properties which defines a pricing rule. That is, $C^E(\cdot)$ is sublinear, arbitrage free, normalized, monotonic and constant additive. In order to prove the DCC property, we first assume that $C(XEk) \geq k$. Then, there is $P_0 \in \mathcal{K}$ satisfying

$$\sum_{s \in E} P_0(s)X(s) + kP_0(E^C) \geq k$$

Since $P_0(E) > 0$ by hypothesis, then $E_{P_0^E}(X) \geq k$. Therefore, $C^E(X) \geq k$.

$(i) \Rightarrow (ii)$. Since $C^E(\cdot)$ is a pricing rule, there is a closed and convex set $\mathcal{L}$ of probabilities on E, where at least one element has full support and

$$C^E(X) = \max_{P \in \mathcal{L}} E_P(X).$$

We must show that $\mathcal{L} = \mathcal{K}^E$. Suppose that there is $P_0 \in \mathcal{L} \setminus \mathcal{K}^E$. By the Separating Hyperplane Theorem, there is X such that $E_{P_0}(X) > E_Q(X)$ for all $Q \in \mathcal{K}^E$. Taking $k = E_{P_0}(X)$, we have that $C^E(X) \geq k$. Therefore, by the DCC property, $C(XEk) \geq k$. On the other hand,

$$C(XEk) = \max_{P \in \mathcal{K}} \left(\sum_{s \in E} P(s)X(s) + kP(E^C) \right) = \sum_{s \in E} P_1(s)X(s) + kP_1(E^C),$$

for some $P_1 \in \mathcal{K}$. Using the hypothesis $P_1(E) > 0$, then $E_{P_1^E}(X) \geq k$. A contradiction, since $k > E_Q(X)$ for all $Q \in \mathcal{K}^E$.

Now, suppose that there is $P_0^E \in \mathcal{K}^E \setminus \mathcal{L}$, where $P_0 \in \mathcal{K}$. By the Separating Hyperplane Theorem, there is X such that $E_{P_0^E}(X) > E_Q(X)$ for all $Q \in \mathcal{L}$. In this case, $k > C^E(X)$, where $k = E_{P_0^E}(X)$. Observe that

$$C(XEk) \geq \sum_{s \in E} P_0(s)X(s) + kP_0(E^C).$$

Then,

$$\frac{C(XEk)}{P_0(E)} \geq \sum_{s \in E} \frac{P_0(s)X(s)}{P_0(E)} + k\frac{P_0(E^C)}{P_0(E)} = k + k\frac{P_0(E^C)}{P_0(E)} = \frac{k}{P_0(E)}.$$

Since $C(XEk) \geq k$ implies $C^E(X) \geq k$ by the DCC property, we have a contradiction. □

In order to become the demonstration of Theorem 6 smoother, we first present some useful results. The first can be found in Jaffray (1992) and shows that the set of conditional risk-neutral probabilities $\mathcal{K}^E$ of a convex and compact set $\mathcal{K}$ is also a convex and compact set.

Lemma 1 *Let $P, Q \in \Delta$ be two probabilities such that $P(E), Q(E) > 0$ and $\alpha \in (0,1)$ a real number. Then, for the real number β defined by*

$$\beta := \left(\frac{P(E)}{Q(E)}\frac{1-\alpha}{\alpha} + 1\right)^{-1},$$

we have $(\beta P + (1-\beta)Q)^E = \alpha P^E + (1-\alpha)Q^E$.

Proof. It is not difficult to verify that

$$\frac{\beta P(F \cap E) + (1-\beta)Q(F \cap E)}{\beta P(E) + (1-\beta)Q(E)} = \alpha\frac{P(F \cap E)}{P(E)} + (1-\alpha)\frac{Q(F \cap E)}{Q(E)},$$

for all events $F \subset \Delta$. Indeed, if $P(F \cap E) = x$, $Q(F \cap E) = y$, $P(E) = z$ and $Q(E) = w$, then $\beta = \frac{\alpha w}{z + \alpha(w-z)}$. Therefore,

$$\frac{\beta P(F \cap E) + (1-\beta)Q(F \cap E)}{\beta P(E) + (1-\beta)Q(E)} = \frac{\frac{\alpha w(x-y)}{z+\alpha(w-z)} + y}{\frac{\alpha w(z-w)}{z+\alpha(w-z)} + w} =$$

$$= \frac{\alpha w(x-y) + zy + \alpha y(w-z)}{\alpha w(z-w) + wz + \alpha w(w-z)} = \frac{\alpha(wx - yz) + zy}{wz} = \alpha\left(\frac{x}{z} - \frac{y}{z}\right) + \frac{y}{z}.$$

Changing back the variables x, y, z, w we have the desired result. □

Using induction over the number of probabilities, it is possible to demonstrate the following important corollary:

Corollary 1 *Let $\mathcal{K} = conv\{P_1, ..., P_m\}$ be the convex hull of the probabilities P_1, P_2,..., P_m. If $P_i(E) > 0$ for all $1 \leq i \leq m$, then $\mathcal{K}^E = conv\{P_1^E, ..., P_m^E\}$.*

Observe that the previous lemma can be extended for values of the parameter α outside the interval $(0, 1)$ and for signed probabilities:

Lemma 2 *Let P, Q be two signed probabilities such that $P(E) \cdot Q(E) \neq 0$ and $P(E) \neq Q(E)$. Let $\alpha \neq \frac{P(E)}{P(E)-Q(E)}$ be a real number. Then, for the real number β defined by*

$$\beta := \left(\frac{P(E)}{Q(E)} \frac{1-\alpha}{\alpha} + 1 \right)^{-1},$$

we have $(\beta P + (1-\beta)Q)^E = \alpha P^E + (1-\alpha)Q^E$.

Proof. Identical to the Lemma 1. In fact, to demonstrate the previous lemma, it was not necessary use the hypothesis of positive probabilities measures. Now, let us constrast useful relations between α and β:

(i) If $P(E) \cdot Q(E) > 0$, then

$$\frac{\partial \beta}{\partial \alpha} = \frac{P(E) \cdot Q(E)}{((1-\alpha)P(E) + \alpha Q(E))^2} > 0.$$

(ii) If $P(E) > Q(E) > 0$ and $\alpha = \lambda \cdot \frac{P(E)}{P(E)-Q(E)}$ with $\lambda \in \left(\frac{P(E)-Q(E)}{P(E)}, 1 \right)$, then $\beta = \frac{\lambda}{1-\lambda} \frac{Q(E)}{P(E)-Q(E)} > 1$ and $\beta P(E) + (1-\beta)Q(E) > 0$.

(iii) If $Q(E) > P(E) > 0$ and $\alpha = \lambda \cdot \frac{P(E)}{Q(E)-P(E)}$ with $\lambda \in \left(\frac{Q(E)-P(E)}{P(E)}, 1 \right)$, then $\beta = \frac{\lambda}{1+\lambda} \frac{Q(E)}{Q(E)-P(E)} > 1$ and $\beta P(E) + (1-\beta)Q(E) > 0$.

The above condition stating that $\beta P(E) + (1-\beta)Q(E) > 0$ for $\alpha \in \left(1, \frac{P(E)}{\|P(E)-Q(E)\|}\right)$ will be useful to proof the Theorem 6. In order to become the demonstration of this theorem smoother, we will present first the following lemma:

Lemma 3 *If $\mathcal{K}$ is a convex polytope and expansible set, there are probabilities $P, Q \in \mathcal{K}$ and a real number $\bar{\alpha} > 1$ such that $\alpha P + (1-\alpha)Q \in \Delta \setminus \mathcal{K}$ for all $\alpha \in (1, \bar{\alpha}]$.*

Proof. Since $\mathcal{K}$ is expansible, there are $Q, R \in \mathcal{K}$ and $\hat{\theta}$ such that $\hat{\theta}R + (1-\hat{\theta})Q \in \Delta \setminus \mathcal{K}$. Now, let $\bar{\theta}$ be the extremal value for the parameter θ in the following sense:

$$\bar{\theta} := \max\{\theta \mid \theta R + (1-\theta)Q \in \mathcal{K}\}. \quad (2)$$

Since $\mathcal{K}$ is a convex polytope, the condition $\theta R + (1-\theta)Q \in \mathcal{K}$ can be expressed as a linear system $A\theta \leq b$. So, the problem (2) has a solution and by linearity this solution is unique. Therefore, $\bar{\theta}$ is well-defined.

Now, observe that

i) $\theta R + (1-\theta)Q \in \mathcal{K}$ for all $\theta \in [1, \bar{\theta}]$;

ii) $\theta R + (1-\theta)Q \in \Delta \setminus \mathcal{K}$ for all $\theta \in (\bar{\theta}, \hat{\theta}]$.

This assertion can be visualized as shown in the Figure 5. Taking $P = \bar{\theta}R + (1-\bar{\theta})Q$ and $\bar{\alpha} = \hat{\theta}/\bar{\theta}$, we have the desired result. □

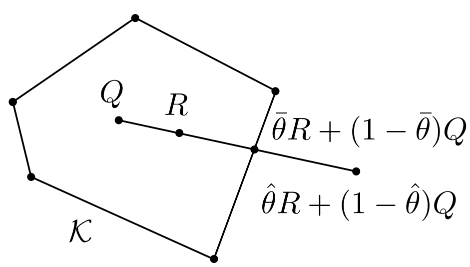

Figure 5: Geometric representation of the Lemma 3

Remark. When $\mathcal{K}$ has at least one strictly positive element, then the probabilities P and Q can be taken strictly positive. ■

Proof of Theorem 6: $(i) \Rightarrow (ii)$: Suppose that $\mathcal{K}$ is a non-expansible polytope and that $\mathcal{K}^E$ is an expansible set. Since $\mathcal{K}$ has at least one strictly positive element, $\mathcal{K}^E$ has a probability on $\Delta(E)^+$. By the Lemma 3, there are $P^E, Q^E \in \mathcal{K}^E$ strictly positive and a parameter $\bar{\alpha} > 1$ such that $\alpha P^E + (1-\alpha)Q^E \in \Delta \setminus \mathcal{K}^E$ for all $\alpha \in (1, \bar{\alpha}]$. By the non trivial updating condition, there are $P, Q \in \mathcal{K}$ be two probabilities such that P^E, Q^E are its respective Bayesian updates and $P(s) > 0$, $Q(s) > 0$ for all $s \in E^C$. Now, we have two cases to be considered:

(i) **Case 1.** If $P(E) = Q(E)$: It is immediate that $\alpha P^E + (1-\alpha)Q^E = (\alpha P + (1-\alpha)Q)^E$. Therefore, $(\alpha P + (1-\alpha)Q)(s) \geq 0$ for all $\alpha \in (1, \bar{\alpha}]$. By the non trivial updating condition, $P(s)$ and $Q(s)$ are positive for every $s \in E^C$. Observe that if $P(s) = Q(s)$ for some $s \in E^C$, then $(\alpha P + (1-\alpha)Q)(s) \geq 0$. Otherwise, if $P(s) < Q(s)$ notice that

$$(\alpha P + (1-\alpha)Q)(s) \geq 0 \iff \alpha \leq \frac{Q(s)}{Q(s) - P(s)},$$

or, if $P(s) > Q(s)$, then

$$(\alpha P + (1-\alpha)Q)(s) \geq 0, \quad \forall \alpha > 0.$$

Thus, there is $\hat{\alpha} > 1$ sufficiently small such that $\alpha P + (1-\alpha)Q \in \Delta$. Since $\mathcal{K}$ is non-expansible, we must have that $\alpha P + (1-\alpha)Q \in \mathcal{K}$, so $\alpha P^E + (1-\alpha)Q^E \in \mathcal{K}^E$. Contradiction.

(ii) **Case 2.** If $P(E) \neq Q(E)$: From the Lemma 2, we know there is a parameter $\beta = \beta(\alpha) > 1$ such that $(\beta P + (1-\beta)Q)^E = \alpha P^E + (1-\alpha)Q^E$ for each $\alpha \in (1, \bar{\alpha}]$. Let us assume without loss of generality that $\bar{\alpha} \in \left(1, \frac{P(E)}{\|P(E)-Q(E)\|}\right)$. Our objective is to ensure that $\beta P + (1-\beta)Q$ is a positive probability for some $\hat{\alpha} > 1$ sufficiently small. We have to consider the following possibilities:

(a) If $s \in E$: We know that

$$\beta P(s)+(1-\beta)Q(s) = \underbrace{[\beta P(E) + (1-\beta)Q(E)]}_{\geq 0} \cdot \underbrace{\left[\alpha \frac{P(s)}{P(E)} + (1-\alpha)\frac{Q(s)}{Q(E)}\right]}_{\geq 0},$$

then $\beta P(s) + (1-\beta)Q(s) \geq 0$.

(b) If $s \in E^C$ and $\frac{P(s)}{P(E)} - \frac{Q(s)}{Q(E)} \geq 0$, the above argument is sufficient to show the non-negativity of $\beta P(s) + (1-\beta)Q(s)$. Otherwise, it is possible to find a small value $\hat{\alpha}$ such that

$$1 \leq \hat{\alpha} < \frac{Q(s)P(E)}{Q(s)P(E) - P(s)Q(E)},$$

for all $s \in E^C$ in which $\frac{P(s)}{P(E)} < \frac{Q(s)}{Q(E)}$. In fact, it is possible assume this constrain because we have that $\frac{Q(s)P(E)}{Q(s)P(E)-P(s)Q(E)} > 1$ by assumption and that $Q(s) > 0$ for all $s \in E^C$ by the non trivial updating condition. Then, for any $\alpha \in (1, \hat{\alpha})$ we have that

$$\alpha \frac{P(s)}{P(E)} + (1-\alpha)\frac{Q(s)}{Q(E)} \geq 0.$$

Therefore, the existence of a positive probability $\beta P + (1-\beta)Q$ with $\beta > 1$ is incompatible with the hypothesis that $\mathcal{K}$ is non-expansible, we must have that $\mathcal{K}^E$ is also a non-expansible set.

(ii) $\Rightarrow$ *(i)*: Suppose that there is $R \in \mathcal{K}^E$ such that there is no $\widetilde{R} \in \mathcal{K}$ where $\widetilde{R}^E = R$ and $\widetilde{R}(s) > 0$ for all $s \in E^C$. Let $P \in \mathcal{K} \cap \Delta(S)^+$ and P^E its Bayesian update. By hypothesis, $P^E \neq R$. Since $\mathcal{K}^E$ is non-expansible and $P^E, R \in \Delta(E)^+$, there is $\bar{\alpha} > 1$ such that $Q = \bar{\alpha}R + (1-\bar{\alpha})P^E \in \mathcal{K}^E$. In this case, $R = \frac{1}{\bar{\alpha}}Q + (1-\frac{1}{\bar{\alpha}})P^E$. Now, since $Q \in \mathcal{K}^E$, there is $\widetilde{Q} \in \mathcal{K}$ such that $\widetilde{Q}^E = Q$. By the Lemma 1, since $\widetilde{Q}(E) > 0$ there is $\beta \in (0,1)$ such that $(\beta\widetilde{Q} + (1-\beta)P)^E = R$. However, $\beta\widetilde{Q} + (1-\beta)P \in \mathcal{K}$ and $(\beta\widetilde{Q} + (1-\beta)P)(s) > 0$ for all $s \in E^C$. A contradiction. □

Proof of Theorem 7: Suppose that a market with uniform bid-ask spreads is represented by the set of risk-neutral measures $\mathcal{K}$ such that

$$\mathcal{K} = (1-\varepsilon)Q + \varepsilon\Delta.$$

Let $P_1, P_2, ..., P_n$ the extremal points of $\mathcal{K}$, that is, $P_i = (1-\varepsilon)Q + \varepsilon\delta_i$. Now, suppose that an event E is observed. In this case, the conditional probabilities of P_i given E are such that

$$P_i^E(s) = \frac{(1-\varepsilon)Q(s)}{(1-\varepsilon)\sum_{s'\in E}(1-\varepsilon)Q(s')} = Q^E(s), \quad \forall s \in E,$$

when $i \notin E$. And

$$P_i^E(s) = \frac{(1-\varepsilon)Q(s)}{(1-\varepsilon)\sum_{s'\in E}(1-\varepsilon)Q(s') + \varepsilon}, \quad \forall s \in E, s \neq i,$$

$$P_i^E(i) = \frac{(1-\varepsilon)Q(i) + \varepsilon}{(1-\varepsilon)\sum_{s'\in E}(1-\varepsilon)Q(s') + \varepsilon},$$

when $i \in E$. Now, observe that for all $i \in E$, $P_i^E = (1-\gamma)Q^E + \gamma\delta_i$, where $\gamma \in (0, 1]$ is a parameter defined as

$$\gamma = \frac{\varepsilon}{(1-\varepsilon)\sum_{s'\in E} Q(s') + \varepsilon}.$$

Therefore, $\mathcal{K}^E$ also represents a market with uniform bid-ask spreads, since $\mathcal{K}^E = (1-\gamma)Q^E + \gamma\Delta^E$. □

Proof of Theorem 8:
$(ii) \Rightarrow (i)$: Since ν is regular, for every nonempty E we have $[\text{acore}(\nu)]^E = \text{acore}(\nu^E)$. Then,

$$C^E(X) = \max_{P\in[\text{acore}(\nu)]^E} E_P(X) = \max_{P\in\text{acore}(\nu^E)} E_P(X).$$

Following Chateauneuf and Jaffray (1995), we know that ν^E must be a concave capacity. Using the the dual version of Theorem 3 in Schmeidler (1986), it is clear that $C^E(X) = \int X d\nu^E$.

$(iii) \Rightarrow (ii)$: It is known from Chateauneuf et al. (2011) that $[\text{acore}(\nu)]^E \subset \text{acore}(\nu^E)$. Suppose that $[\text{acore}(\nu)]^E \subsetneq \text{acore}(\nu^E)$ for some event $E \neq S$. Then, there is a probability $P_0 \in \text{acore}(\nu^E) \setminus [\text{acore}(\nu)]^E$. From Jaffray (1992) we know that $[\text{acore}(\nu)]^E$ is a convex compact set. By the strong version of the Separating Hyperplane Theorem (see Rockafellar (1997)), there is $X \in \mathbb{R}^S$ such that $E_{P_0}(X) > E_P(X)$ for every $P \in [\text{acore}(\nu)]^E$. In the other side, the hypothesis ensures that $C^E(X) = \max_{P\in[\text{acore}(\nu)]^E} E_P(X) = \max_{P\in\text{acore}(\nu^E)} E_P(X)$ for every $X \in \mathbb{R}^S$, a contradiction.

Using the same argument, it is possible to show that $[\text{acore}(\nu)]^E \supsetneq \text{acore}(\nu^E)$ cannot be true. Therefore, $[\text{acore}(\nu)]^E = \text{acore}(\nu^E)$.

$(i) \Rightarrow (iii)$: First, take $E = S$. Then $C(X) = \max_{P \in \mathcal{K}}$ is a Choquet integral for some capacity ν. Furthermore, $C(A^*) = \nu(A)$ for all $A \subset S$. Therefore, ν is unique.

Since $C(\cdot)$ is subadditive, we obtain that

$$\int (X+Y) d\nu \leq \int X d\nu + \int Y d\nu,$$

for all $X, Y \in \mathbb{R}^S$. From the dual version of Theorem 3 in Schmeidler (1986), we conclude that ν must be concave. Moreover, $C(X) = \max\limits_{P \in \mathcal{K}} E_P(X) = \max\limits_{P \in \text{acore}(\nu)} E_P(X)$. Then, using again the strong version of the Separating Hyperplane Theorem, we have that $\mathcal{K} = \text{acore}(\nu)$.

Suppose that $C^E(\cdot)$ is a Choquet integral with respect to a concave capacity μ_E. Since ν is concave, we know from Chateauneuf et al. (2011) that $\nu^E(A) = \sup\{P^E(A) \mid P \in \text{acore}(\nu)\}, \forall A \subset E$ can also be written as

$$\nu^E(A) = \frac{\nu(A)}{\nu(A) + 1 - \nu(A \cup E^C)},$$

and that ν^E is concave for every event E compatible with $\mathcal{K}$. And by the unicity in the representation, we must have $\mu_E = \nu^E$. This completes the proof. □

References

Araujo, A., Chateauneuf, A., and Faro, J. (2012). Pricing rules and Arrow-Debreu ambiguous valuation. *Economic Theory*, 49:1–35.

Araujo, A., Chateauneuf, A., and Faro, J. (2015). Financial Market Structures Revealed by Pricing Rules: Efficient Complete Markets are Prevalent. *Insper Working Paper*.

Cerreia-Vioglio, S., Maccheroni, F., and Marinacci, M. (2015). Put-call parity and market frictions. *Journal of Economic Theory*, 157:730–762.

Chateauneuf, A., Gajdos, T., and Jaffray, J.-Y. (2011). Regular updating. *Theory and Decision*, 71:111–128.

Chateauneuf, A. and Jaffray, J.-Y. (1995). Local mbius transforms of monotone capacities. In *Symbolic and Quantitative Approaches to Reasoning and Uncertainty*, pages 115–124. Springer Berlin Heidelberg.

Chateauneuf, A., Kast, R., and Lapied, A. (1996). Choquet Pricing For Financial Markets With Frictions. *Mathematical Finance*, 6:323–330.

Cox, J. C. and Ross, S. A. (1976). The valuation of options for alternative stochastic processes. *Journal of Financial Economics*, 3:145–166.

Fagin, R. and Halpern, J. Y. (1990). A new approach to updating beliefs. In *Proceedings of the Sixth Annual Conference on Uncertainty in Artificial Intelligence*.

Faro, J. H. and Lefort, J. P. (2013). Dynamic Objective and Subjective Rationality. Technical report, *Insper Working Paper WPE: 312/2013.*

Galanis, S. (2014). Dynamic consistency and subjective beliefs. Technical report, mimeo.

Huber, P. J. (1981). *Robust Statistics*. Wiley Series in Probabilities and Mathematical Statistics, John Wiley & Sons.

Jaffray, J.-Y. (1992). Bayesian updating and belief functions. *IEEE Transactions on Systems, Man, and Cybernetics*, 22:1144–1152.

Jouini, E. and Kallal, H. (1995). Martingales and Arbitrage in Securities Markets with Transaction Costs. *Journal of Economic Theory*, 66:178–197.

Luttmer, E. G. J. (1996). Asset pricing in economies with frictions. *Econometrica*, 64(6):1439–1467.

Pires, C. P. (2002). A rule for updating ambiguous beliefs. *Theory and Decision*, 53:137–152.

Rockafellar, R. (1997). *Convex Analysis*. Princeton University Press.

Ross, S. A. (1976). The arbitrage theory of capital asset pricing. *Journal of economic theory*, 13:341–360.

Ross, S. A. (1978). A simple approach to the valuation of risky streams. *Journal of business*, 51:453–475.

Schmeidler, D. (1986). Integral representation without additivity. *Proceedings of the American Mathematical Society*, 97:255–261.

Schneider, R. (1993). *Convex Bodies: The Brunn-Minkowski Theory*. Cambridge University Press.

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